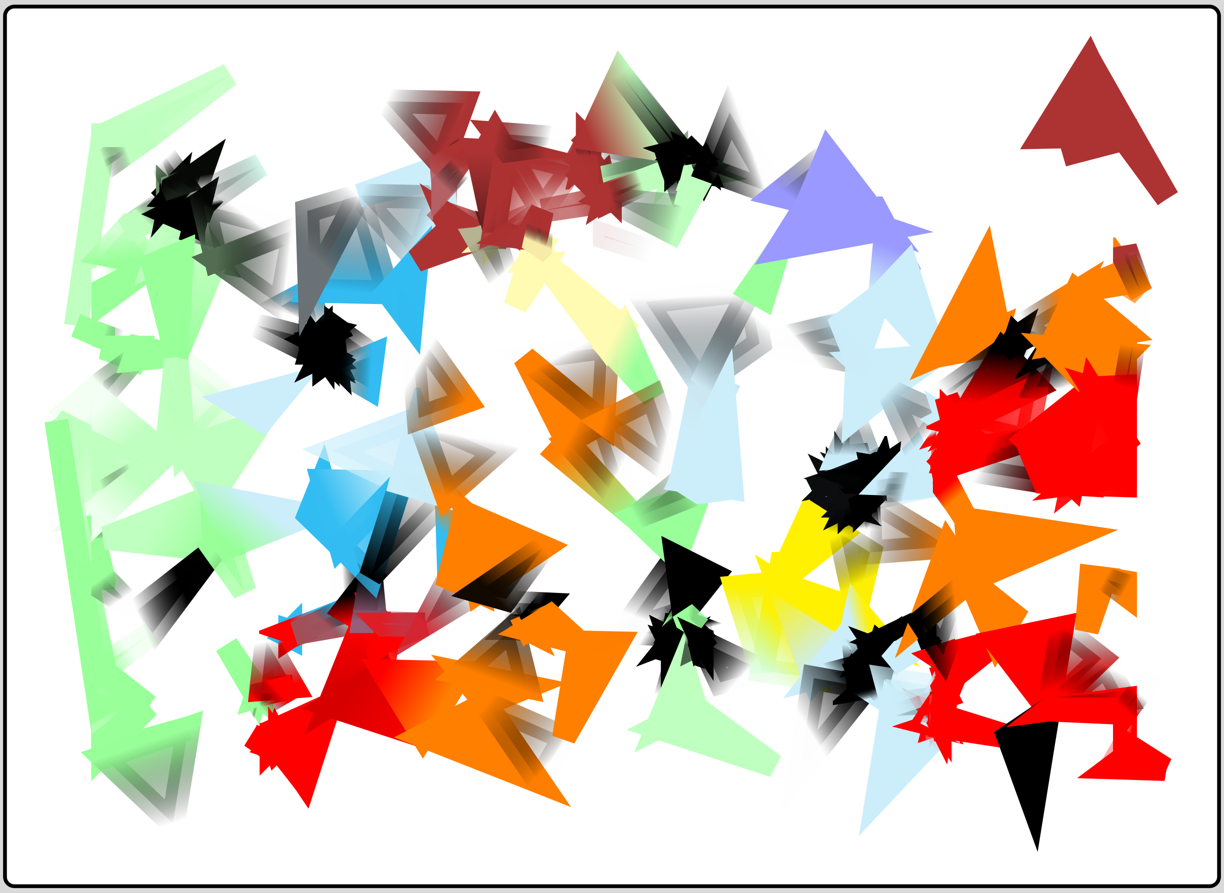


Some Olympiad Mathematics Problems arranged by MOHS Hardness Scale

December 1, 2020

Introduction

Evan Chen has introduced a "MOHS hardness scale" for rating the difficulty of mathematics contest problems in his blog:

https://usamo.wordpress.com/2019/11/26/mohs-hardness-scale/

The scales are from 0M to roughly 50M, with intermediate steps being 5. The higher the scale, the more difficult the problem is. MOHS has became popular among AoPS community and you can find numerous discussions on how hard a problem is and what is its MOHS every now and then.

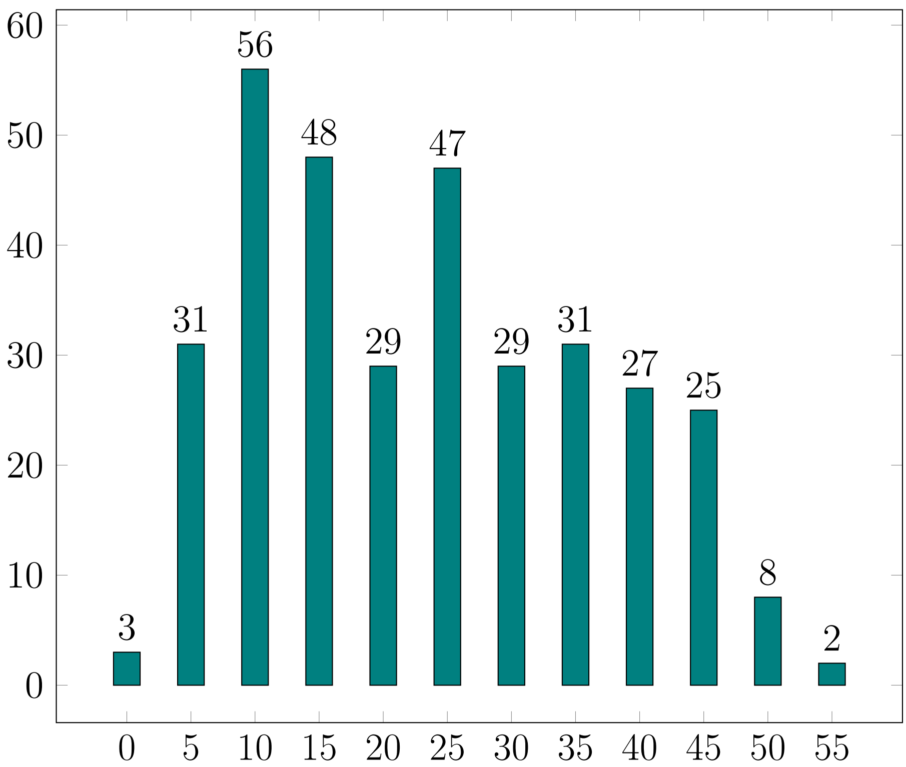
This collection contains 336 problems on Algebra, Combinatorics, Geometry and Number Theory of some IMO, USAMO, USA TSTST and USA TST arranged by their MOHS in increasing order, according to the index available here:

https://web.evanchen.cc/upload/MOHS-hardness.pdf

We use the following AoPS resources for this collection:

IMO Collection, USAMO Collection, USA TSTST, USA TST

The distribution of problems is depicted in the diagram below:



We hope this collection will have future updates and will be beneficial for mathematics contests training purposes.

Rustam Turdibaev and Olimjon Olimov

Contents

1 Algebra

1.1 0 M

1. (USAMO 2012 P1) Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \ldots, a_n$ with $\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n)$, there exist three that are the side lengths of an acute triangle.

AoPS discussion thread

1.2 5 M

2. (IMO 2000, P2) Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

AoPS discussion thread

3. (IMO 2001, P4) Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

AoPS discussion thread

4. (IMO 2007, P1) Real numbers a_1, a_2, …, a_n are given. For each i, $(1 \leq i \leq n)$, define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let $d = \max\{d_i \mid 1 \leq i \leq n\}$.

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that the equality holds in (*).

AoPS discussion thread

5. (IMO 2014, P1) Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

AoPS discussion thread

6. (IMO 2019, P1) Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

$\boxed{\text{AoPS discussion thread}}$

7. (USAMO 2002 P2) Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

$\boxed{\text{AoPS discussion thread}}$

8. (USA TST 2014, P4) Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

$\boxed{\text{AoPS discussion thread}}$

1.3 10 M

9. (IMO 2004, P4) Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

$\boxed{\text{AoPS discussion thread}}$

10. (IMO 2008, P4) Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

AoPS discussion thread

11. (IMO 2010, P1) Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

AoPS discussion thread

12. (USAMO 2000 P1) Call a real-valued function f very convex if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x - y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

AoPS discussion thread

13. (USAMO 2000 P2) Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP} + \frac{1}{BQ} + \frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

AoPS discussion thread

14. (USAMO 2002 P4) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

AoPS discussion thread

15. (USAMO 2006 P2) For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

AoPS discussion thread

16. (USAMO 2006 P4) Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \ldots + a_k = a_1 \cdot a_2 \cdots a_k = n$.

AoPS discussion thread

17. (USAMO 2007 P1) Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k - 1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any n the sequence $a_1, a_2, \ldots$ eventually becomes constant.

AoPS discussion thread

18. (USAMO 2009 P4) For $n \ge 2$ let $a_1, a_2, \ldots a_n$ be positive real numbers such that
$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \le \left(n + \frac{1}{2}\right)^2.$$
Prove that $\max(a_1, a_2, \ldots, a_n) \le 4\min(a_1, a_2, \ldots, a_n)$.

AoPS discussion thread

19. (USAMO 2011 P1) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \le 4$. Prove that
$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

AoPS discussion thread

20. (USAMO 2013 P4) Find all real numbers $x, y, z \ge 1$ satisfying
$$\min(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

AoPS discussion thread

21. (USAMO 2014 P1) Let a, b, c, d be real numbers such that $b - d \ge 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

AoPS discussion thread

22. (USAMO 2018 P1) Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that
$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \ge a^2 + b^2 + c^2.$$

AoPS discussion thread

23. (USA TST 2015, P4) Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x+y) - f(x) - f(y)$ is an integer. Decide whether it follows that there exists a constant c such that $f(x) - cx$ is an integer for every rational number x.

AoPS discussion thread

24. (USA TSTST 2015, P1) Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let m be a fixed positive integer less than n. We say an index k with $1 \leq k \leq n$ is good if there exists some ℓ with $1 \leq \ell \leq m$ such that $a_k + a_{k+1} + ... + a_{k+\ell-1} \geq 0$, where the indices are taken modulo n. Let T be the set of all good indices. Prove that $\sum\limits_{k \in T} a_k \geq 0$.

AoPS discussion thread

1.4 15 M

25. (IMO 2002, P5) Find all functions f from the reals to the reals such that

$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

AoPS discussion thread

26. (IMO 2005, P2) Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n.

Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

AoPS discussion thread

27. (IMO 2009, P5) Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

AoPS discussion thread

28. (IMO 2012, P2) Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

AoPS discussion thread

29. (IMO 2012, P4) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

AoPS discussion thread

30. (USAMO 2003 P5) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

AoPS discussion thread

31. (USAMO 2012 P4) Find all functions $f:\mathbb{Z}^+\to\mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!)=f(n)!$ for all positive integers n and such that $m-n$ divides $f(m)-f(n)$ for all distinct positive integers m,n.

AoPS discussion thread

32. (USAMO 2019 P1) Let $\mathbb{N}$ be the set of positive integers. A function $f:\mathbb{N}\to\mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n)\text{ times}}(n)\ldots))=\frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

AoPS discussion thread

33. (USA TST 2020, P1) Choose positive integers $b_1,b_2,\ldots$ satisfying

$$1=\frac{b_1}{1^2}>\frac{b_2}{2^2}>\frac{b_3}{3^2}>\frac{b_4}{4^2}>\cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2}\ge r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

AoPS discussion thread

34. (USA TSTST 2014, P4) Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ such that:

    (i) both A and B have degree at most $d/2$

    (ii) at most one of A and B is the zero polynomial.

    (iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x)+Q(x)B(x)=P(x)C(x)$.

AoPS discussion thread

1.5 20 M

35. (IMO 2001, P2) Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

AoPS discussion thread

36. (IMO 2004, P2) Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab+bc+ca=0$ we have the following relations

$$f(a-b)+f(b-c)+f(c-a)=2f(a+b+c).$$

AoPS discussion thread

37. (IMO 2008, P2) (a) Prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz=1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz=1$.

AoPS discussion thread

38. (USAMO 2018 P2) Find all functions $f:(0,\infty)\to(0,\infty)$ such that

$$f\left(x+\frac{1}{y}\right)+f\left(y+\frac{1}{z}\right)+f\left(z+\frac{1}{x}\right)=1$$

for all $x, y, z>0$ with $xyz=1$.

AoPS discussion thread

39. (USA TSTST 2019, P1) Find all binary operations $\diamondsuit:\mathbb{R}_{>0}\times\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c>0$,

- the equation $a\diamondsuit(b\diamondsuit c)=(a\diamondsuit b)\cdot c$ holds; and
- if $a\geq 1$ then $a\diamondsuit a\geq 1$.

AoPS discussion thread

1.6 25 M

40. (IMO 2003, P5) Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

AoPS discussion thread

41. (IMO 2013, P5) Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

    (i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

    (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

    (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

    Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

AoPS discussion thread

42. (IMO 2016, P5) The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

AoPS discussion thread

43. (IMO 2020, P2) The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1$$

AoPS discussion thread

44. (USAMO 2003 P3) Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

AoPS discussion thread

45. (USAMO 2004 P5) Let $a, b, c > 0$. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

AoPS discussion thread

46. (USAMO 2014 P2) Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

AoPS discussion thread

47. (USAMO 2016 P4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

AoPS discussion thread

48. (USAMO 2019 P6) Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x - y) + P(y - z) + P(z - x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

AoPS discussion thread

49. (USA TST 2016, P5) Let $n \geq 4$ be an integer. Find all functions $W : \{1, \ldots, n\}^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a, b) W(b, c) = |A||B||C|.$$

AoPS discussion thread

50. (USA TSTST 2014, P3) Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

for all real x with $|x| \le 1$.

AoPS discussion thread

51. (USA TSTST 2016, P1) Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

AoPS discussion thread

52. (USA TSTST 2017, P3) Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

AoPS discussion thread

1.7 30 M

53. (IMO 2005, P3) Let x, y, z be three positive reals such that $xyz \ge 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \ge 0.$$

AoPS discussion thread

54. (IMO 2015, P6) The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

(i) $1 \le a_j \le 2015$ for all $j \ge 1$, (ii) $k + a_k \ne \ell + a_\ell$ for all $1 \le k < \ell$.

Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \le 1007^2$$

for all integers m and n such that $n > m \ge N$.

AoPS discussion thread

55. (IMO 2018, P2) Find all integers $n \ge 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

AoPS discussion thread

56. (USAMO 2001 P3) Let $a, b, c \geq 0$ and satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$0 \leq ab + bc + ca - abc \leq 2.$$

AoPS discussion thread

57. (USAMO 2002 P3) Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

AoPS discussion thread

58. (USAMO 2014 P3) Prove that there exists an infinite set of points

$$\ldots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \ldots$$

in the plane with the following property:

> For any three distinct integers $a, b,$ and c, points P_a, P_b, and P_c are collinear if and only if $a + b + c = 2014$.

AoPS discussion thread

59. (USAMO 2015 P6) Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$. (A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equivalent, but $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ differ.)

AoPS discussion thread

60. (USAMO 2017 P6) Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4}$$

given that a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$.

AoPS discussion thread

61. (USA TST 2018, P2) Find all functions $f : \mathbb{Z}^2 \to [0,1]$ such that for any integers x and y,

$$f(x,y) = \frac{f(x-1,y) + f(x,y-1)}{2}.$$

AoPS discussion thread

62. (USA TSTST 2015, P4) Let x, y, and z be real numbers (not necessarily positive) such that

$$x^4 + y^4 + z^4 + xyz = 4.$$

Show that $x \le 2$ and $\sqrt{2-x} \ge \frac{y+z}{2}$.

AoPS discussion thread

63. (USA TSTST 2017, P6) A sequence of positive integers $(a_n)_{n\ge 1}$ is of Fibonacci type if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \ge 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

AoPS discussion thread

64. (USA TSTST 2018, P6) Let $S = \{1, \ldots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \ldots, a_n) \in S^n \mid a_1 + \cdots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.

AoPS discussion thread

1.8 35 M

65. (IMO 2001, P6) Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

AoPS discussion thread

66. (IMO 2002, P3) Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

AoPS discussion thread

67. (IMO 2006, P3) Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \le M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

AoPS discussion thread

68. (IMO 2015, P5) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x+f(x+y))+f(xy) = x+f(x+y)+yf(x)$$

for all real numbers x and y.

AoPS discussion thread

1.9 40 M

69. (IMO 2009, Problem P3) Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

AoPS discussion thread

70. (IMO 2010, Problem P3) Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m)+n)\,(g(n)+m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

AoPS discussion thread

71. (IMO 2010, Problem P6) Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \le s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \ge N.$$

AoPS discussion thread

72. (IMO 2011, Problem P3) Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

AoPS discussion thread

73. (IMO 2017, Problem P2) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

AoPS discussion thread

74. (USA TST 2017, P3) Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

AoPS discussion thread

1.10 45 M

75. (IMO 2012, P6) Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

AoPS discussion thread

76. (USAMO 2004 P6) A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

AoPS discussion thread

77. (USAMO 2020 P3) Let p be an odd prime. An integer x is called a quadratic non-residue if p does not divide $x - t^2$ for any integer t.

Denote by A the set of all integers a such that $1 \leq a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

AoPS discussion thread

78. (USAMO 2020 P6) Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be $2n$ real numbers such that

$$0 = x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$$

$$\text{and } 1 = x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+i-1}) \geq \frac{2}{\sqrt{n-1}}.$$

AoPS discussion thread

1.11 50 M

79. (IMO 2007, P6) Let n be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

AoPS discussion thread

80. (USAMO 2000 P6) Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

AoPS discussion thread

2 Combinatorics

2.1 0 M

1. (USAMO 2005 P4) Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?

   (The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

   AoPS discussion thread

2.2 5 M

2. (IMO 2002 P1) Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is colored red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

   AoPS discussion thread

3. (IMO 2011, P4) Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.

   Determine the number of ways in which this can be done.

   AoPS discussion thread

4. (USAMO 2002 P1) Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

   (a) the union of any two white subsets is white;

   (b) the union of any two black subsets is black;

   (c) there are exactly N white subsets.

   AoPS discussion thread

5. (USAMO 2005 P1) Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

AoPS discussion thread

6. (USAMO 2019 P4) Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \le i, j \le n$, such that:

   - for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and
   - $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

AoPS discussion thread

2.3 10 M

7. (IMO 2003, P1) Let A be a 101-element subset of the set

$$S = \{1, 2, \ldots, 1000000\}.$$

Prove that there exist numbers t_1, $t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

AoPS discussion thread

8. (USAMO 2002 P5) Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \le i < k)$.

AoPS discussion thread

9. (USAMO 2004 P4) Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

AoPS discussion thread

10. (USAMO 2005 P5) Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

AoPS discussion thread

11. (USAMO 2008 P4) Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

AoPS discussion thread

12. (USAMO 2011 P6) Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \ldots \cup A_{11}| \geq 165$, and give an example for which equality holds.

AoPS discussion thread

13. (USAMO 2017 P4) Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

AoPS discussion thread

14. (USAMO 2018 P4) Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

AoPS discussion thread

15. (USA TST 2015, P5) A tournament is a directed graph for which every (unordered) pair of vertices has a single directed edge from one vertex to the other. Let us define a proper directed-edge-coloring to be an assignment of a color to every (directed) edge, so that for every pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vw}$, those two edges are in different colors. Note that it is permissible for $\overrightarrow{uv}$ and $\overrightarrow{uw}$ to be the same color. The directed-edge-chromatic-number of a tournament is defined to be the minimum total number of colors that can be used in order to create a proper directed-edge-coloring. For each n, determine the minimum directed-edge-chromatic-number over all tournaments on n vertices.

AoPS discussion thread

16. (USA TST 2016, P1) Let $S = \{1, \ldots, n\}$. Given a bijection $f : S \to S$ an orbit of f is a set of the form $\{x, f(x), f(f(x)), \ldots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

    Given k bijections f_1, ..., f_k from S to itself, prove that

    $$c(f_1) + \cdots + c(f_k) \leq n(k-1) + c(f)$$

    where $f : S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

AoPS discussion thread

17. (USA TSTST 2014, P1) Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is reachable from a string A if it is possible to insert some amount of $\leftarrow$'s in A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

    Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

AoPS discussion thread

18. (USA TSTST 2019, P8) Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(S)$ denote the number of ways to draw 8 lines with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

AoPS discussion thread

2.4 15 M

19. (IMO 2000, P4) A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

    How many ways are there to put the cards in the three boxes so that the trick works?

    AoPS discussion thread

20. (IMO 2008, P5) Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

    Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

    Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

    Determine $\frac{N}{M}$.

    AoPS discussion thread

21. (IMO 2014, P2) Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

    AoPS discussion thread

22. (IMO 2015, P1) We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

    (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of

n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

AoPS discussion thread

23. (IMO 2018, P4) A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

AoPS discussion thread

24. (IMO 2019, P5) The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

AoPS discussion thread

25. (IMO 2020, P4) There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start

from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

AoPS discussion thread

26. (USAMO 2000 P3) A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand.

    Find, as a function of $R, W,$ and B, the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

    AoPS discussion thread

27. (USAMO 2000 P4) Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

    AoPS discussion thread

28. (USAMO 2001 P1) Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

    AoPS discussion thread

29. (USAMO 2011 P2) An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

    AoPS discussion thread

30. (USAMO 2014 P4) Let k be a positive integer. Two players A and B play

a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

AoPS discussion thread

31. (USAMO 2015 P4) Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k),(i,l),(j,k),(j,l)$ for some $1 \leq i,j,k,l \leq n$, such that $i<j$ and $k<l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively,j or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

    Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

    How many different non-equivalent ways can Steve pile the stones on the grid?

AoPS discussion thread

32. (USAMO 2016 P1) Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

AoPS discussion thread

33. (USAMO 2016 P6) Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard.

    The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order.

    You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is winnable if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds.

For which values of n and k is the game winnable?

AoPS discussion thread

34. (USA TST 2017 P1) In a sports league, each team uses a set of at most t signature colors. A set S of teams is color-identifiable if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

    For all positive integers n and t, determine the maximum integer $g(n, t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n, t)$.

    AoPS discussion thread

35. (USA TST 2018 P4) Let n be a positive integer and let $S \subseteq \{0, 1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \ldots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their majority is defined as the binary string $y \in \{0, 1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \ldots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

    Suppose that for some positive integer k, S has the property P_k that the majority of any $2k + 1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

    AoPS discussion thread

36. (USA TSTST 2017, P2) Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. (The word does not need to be a valid English word.) Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses.

    For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word.

    Which words can Ana pick so that she wins no matter what value of k Banana chooses?

(The subsequences of a string of length n are the 2^n strings which are formed by deleting some of its characters, possibly all or none, while preserving the order of the remaining characters.)

AoPS discussion thread

37. (USA TSTST 2018, P2) In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

    We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

AoPS discussion thread

38. (USA TSTST 2019, P4) Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we have any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

AoPS discussion thread

2.5 20 M

39. (IMO 2013, P2) A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

    i) No line passes through any point of the configuration.

    ii) No region contains points of both colors.

    Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

AoPS discussion thread

40. (USAMO 2010 P2) There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the

two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

AoPS discussion thread

41. (USAMO 2012 P6) For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \ldots + x_n = 0, \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)

Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

AoPS discussion thread

42. (USAMO 2020 P2) An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

AoPS discussion thread

43. (USAMO 2020 P4) Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

AoPS discussion thread

44. (USA TST 2020, P4) For a finite simple graph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G.

    Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

    AoPS discussion thread

45. (USA TSTST 2014, P5) Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there is no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

    AoPS discussion thread

46. (USA TSTST 2016, P3) Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

    produce at most $0.499n$ distinct residues when taken modulo n.

    AoPS discussion thread

2.6 25 M

47. (IMO 2001, P3) Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

    AoPS discussion thread

48. (IMO 2016, P6) There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

    (a) Prove that Geoff can always fulfill his wish if n is odd.

(b) Prove that Geoff can never fulfill his wish if n is even.

AoPS discussion thread

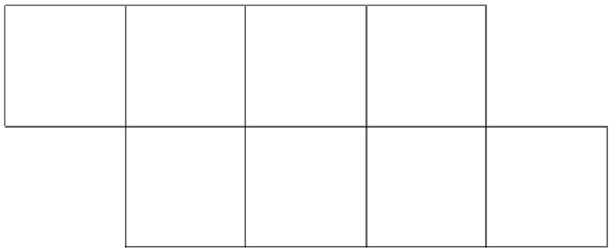
49. (USAMO 2002 P6) I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

AoPS discussion thread

50. (USAMO 2007 P4) An animal with n cells is a connected figure consisting of n equal-sized cells. The figure below shows an 8-cell animal.



A dinosaur is an animal with at least 2007 cells. It is said to be primitive it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

Note: Animals are also called polyominoes. They can be defined inductively. Two cells are adjacent if they share a complete edge. A single cell is an animal, and given an animal with n cells, one with $n+1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

AoPS discussion thread

51. (USAMO 2008 P5) Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

AoPS discussion thread

52. (USAMO 2009 P2) Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \dots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a+b+c=0$.

AoPS discussion thread

53. (USAMO 2012 P2) A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

AoPS discussion thread

54. (USAMO 2013 P2) For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

AoPS discussion thread

55. (USAMO 2015 P3) Let $S = \{1, 2, \dots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

    Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

    $$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

AoPS discussion thread

56. (USA TST 2015, P3) A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons

that she is sure are currently in the same state. Is there any series of diode usage that makes this possible?

AoPS discussion thread

57. (USA TSTST 2016, P5) n the coordinate plane are finitely many walls; which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

    Prove that it is impossible for the bulldozer to hit both sides of every wall.

    AoPS discussion thread

58. (USA TSTST 2018, P7) Let n be a positive integer. A frog starts on the number line at 0. Suppose it makes a finite sequence of hops, subject to two conditions:

    - The frog visits only points in $\{1, 2, \ldots, 2^n - 1\}$, each at most once.
    - The length of each hop is in $\{2^0, 2^1, 2^2, \ldots\}$. (The hops may be either direction, left or right.)

    Let S be the sum of the (positive) lengths of all hops in the sequence. What is the maximum possible value of S?

    AoPS discussion thread

2.7 30 M

59. (IMO 2006, P2) Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

    Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

    AoPS discussion thread

60. (IMO 2014, P5) For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

    AoPS discussion thread

61. (2016, P2) Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that: in each row and each column, one third of the entries are I, one third are M and one third are O; and in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O. Note. The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.

AoPS discussion thread

62. (USAMO 2020 P5) A finite set S of points in the coordinate plane is called overdetermined if $|S| \geq 2$ and there exists a nonzero polynomial $P(t)$, with real coefficients and of degree at most $|S| - 2$, satisfying $P(x) = y$ for every point $(x, y) \in S$.

For each integer $n \geq 2$, find the largest integer k (in terms of n) such that there exists a set of n distinct points that is not overdetermined, but has k overdetermined subsets.

AoPS discussion thread

63. (USA TST 2017, P4) You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestants' guesses before writing down your own. For each question, after all guesses are submitted, the emcee announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, since you hacked the scoring system. After announcing the correct answer, the emcee proceeds to read the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

AoPS discussion thread

64. (USA TST 2019, P5) Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

- Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
- Stacy must remove three consecutive beads which are sapphire, turquoise,

and sapphire, in that order, on each of her turns.

They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

AoPS discussion thread

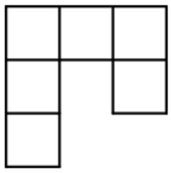
2.8 35 M

65. (IMO 2000, P3) Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

    Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

    AoPS discussion thread

66. (IMO 2004, P3) Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.

    Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

    - the rectangle is covered without gaps and without overlaps

    - no part of a hook covers area outside the rectangle.

    AoPS discussion thread

67. (IMO 2005, P6) In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

    AoPS discussion thread

68. (IMO 2010, P5) Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

AoPS discussion thread

69. (IMO 2014, P6) A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

    Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

    AoPS discussion thread

70. (IMO 2017, P5) An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

    (1) no one stands between the two tallest players,

    (2) no one stands between the third and fourth tallest players,

    $\vdots$

    (N) no one stands between the two shortest players.

    Show that this is always possible.

    AoPS discussion thread

71. (USAMO 2001 P6) Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

    AoPS discussion thread

72. (USAMO 2003 P6) At the vertices of a regular hexagon are written six non-negative integers whose sum is 2003^{2003}. Bert is allowed to make moves of

the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

AoPS discussion thread

73. (USAMO 2004 P3) For what real values of $k > 0$ is it possible to dissect a $1 \times k$ rectangle into two similar, but noncongruent, polygons?

AoPS discussion thread

74. (USAMO 2006 P5) A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

AoPS discussion thread

75. (USAMO 2007 P3) Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

AoPS discussion thread

76. (USAMO 2008 P3) Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that
$$|x| + \left|y + \frac{1}{2}\right| < n.$$
A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

AoPS discussion thread

77. (USA TST 2014, P3) Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge).

An unordered pair of distinct vertices $\{x, y\}$ is said to be amicable if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

AoPS discussion thread

2.9 40 M

78. (IMO 2017, P3) A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order:

    - The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.
    - A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.
    - The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

    Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

    AoPS discussion thread

79. (IMO 2019, P3) A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

    > Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

    Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

    AoPS discussion thread

80. (IMO 2020, P3) There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

    - The total weights of both piles are the same.
    - Each pile contains two pebbles of each colour.

    AoPS discussion thread

81. (USAMO 2008 P6) At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

    AoPS discussion thread

82. (USAMO 2017 P2) Let $m_1, m_2, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

    - $a_i \geq w_i > w_j$
    - $w_j > a_i \geq w_i$, or
    - $w_i > w_j > a_i$.

    Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

    AoPS discussion thread

2.10 45 M

83. (IMO 2006, P6) Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

    AoPS discussion thread

84. (IMO 2009, P6) Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$.

A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

AoPS discussion thread

85. (IMO 2011, P2) Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

AoPS discussion thread

86. (IMO 2012, P3) The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

    At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

    After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

    1. If $n \geq 2^k$, then B can guarantee a win.
    2. For all sufficiently large k, there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.

AoPS discussion thread

87. (IMO 2013, P6) Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points

with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

AoPS discussion thread

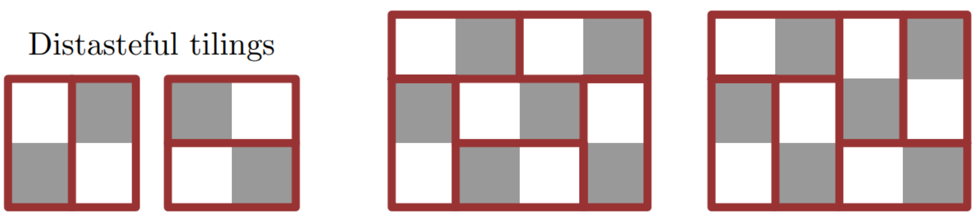
88. (IMO 2018, P3) An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \cdots + 2018$?

AoPS discussion thread

89. (USAMO 2009 P3) We define a chessboard polygon to be a polygon whose sides are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a tasteful tiling is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.

b) Prove that such a tasteful tiling is unique.

AoPS discussion thread

90. (USAMO 2010 P6) A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

AoPS discussion thread

91. (USAMO 2018 P6) Let a_n be the number of permutations $(x_1, x_2, \ldots, x_n)$ of the numbers $(1, 2, \ldots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \le k \le n$ are all distinct. Prove that a_n is odd for all $n \ge 1$.

AoPS discussion thread

92. (USA TST 2018, P3) At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

AoPS discussion thread

93. (USA TST 2018, P6) Alice and Bob play a game. First, Alice secretly picks a finite set S of lattice points in the Cartesian plane. Then, for every line ℓ in the plane which is horizontal, vertical, or has slope $+1$ or -1, she tells Bob the number of points of S that lie on ℓ. Bob wins if he can determine the set S.

Prove that if Alice picks S to be of the form

$$S = \{(x, y) \in \mathbb{Z}^2 \mid m \le x^2 + y^2 \le n\}$$

for some positive integers m and n, then Bob can win. (Bob does not know in advance that S is of this form.)

AoPS discussion thread

94. (USA TST 2020, P3) Let $\alpha \geq 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be flooded. Hephaestus is building a levee, which is a subset of unit edges of the grid (called walls) forming a connected, non-self-intersecting path or loop*.

    The game then begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well. Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

    * More formally, there must exist lattice points $A_0, A_1, \ldots, A_k$, pairwise distinct except possibly $A_0 = A_k$, such that the set of walls is exactly $A_0A_1, A_1A_2, \ldots, A_{k-1}A_k$. Once a wall is built it cannot be destroyed; in particular, if the levee is a closed loop (i.e. $A_0 = A_k$) then Hephaestus cannot add more walls. Since each wall has length 1, the length of the levee is k.

    AoPS discussion thread

95. (USA TSTST 2018, P9) Show that there is an absolute constant $c < 1$ with the following property: whenever $\mathcal{P}$ is a polygon with area 1 in the plane, one can translate it by a distance of $\frac{1}{100}$ in some direction to obtain a polygon $\mathcal{Q}$, for which the intersection of the interiors of $\mathcal{P}$ and $\mathcal{Q}$ has total area at most c.

    AoPS discussion thread

96. (USA TSTST 2019, P3) On an infinite square grid we place finitely many cars, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a move, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

    AoPS discussion thread

2.11 50 M

97. (IMO 2007, P3) In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size.

    Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

    AoPS discussion thread

98. (IMO 2020, P6) Prove that there exists a positive constant c such that the following statement is true: Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

    (A line ℓ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

    Note. Weaker results with $cn^{-1/3}$ replaced by $cn^{-\alpha}$ may be awarded points depending on the value of the constant $\alpha > 1/3$.

    AoPS discussion thread

99. (USAMO 2013 P3) Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An operation is to choose a line parallel to the sides of the triangle, and flipping all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

    AoPS discussion thread

100. (USAMO 2017 P5) Let $\mathbb{Z}$ denote the set of all integers. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points $(x, y) \in \mathbb{Z}^2$ with positive integers for which:

    - only finitely many distinct labels occur, and

- for each label i, the distance between any two points labeled i is at least c^i.

AoPS discussion thread

101. (USA TST 2019, P3) A snake of length k is an animal which occupies an ordered k-tuple $(s_1, \ldots, s_k)$ of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. If the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead. The snake has turned around if it occupied $(s_1, s_2, \ldots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \ldots, s_1)$ instead.

    Determine whether there exists an integer $n > 1$ such that: one can place some snake of length $0.9n^2$ in an $n \times n$ grid which can turn around.

AoPS discussion thread

2.12 55 M

102. (USA TSTST 2015, P6) A Nim-style game is defined as follows. Two positive integers k and n are specified, along with a finite set S of k-tuples of integers (not necessarily positive). At the start of the game, the k-tuple $(n, 0, 0, ..., 0)$ is written on the blackboard. A legal move consists of erasing the tuple $(a_1, a_2, ..., a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, ..., a_k + b_k)$, where $(b_1, b_2, ..., b_k)$ is an element of the set S. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw. Prove that there is a choice of k and S with the following property: the first player has a winning strategy if n is a power of 2, and otherwise the second player has a winning strategy.

AoPS discussion thread

3 Geometry

3.1 5 M

1. (IMO 2004, P1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

   AoPS discussion thread

2. (IMO 2006, P1) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

   $$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

   Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

   AoPS discussion thread

3. (IMO 2007, P4) In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

   AoPS discussion thread

4. (IMO 2012, P1) Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

   (The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

   AoPS discussion thread

5. (IMO 2014, P4) Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC.

   AoPS discussion thread

6. (USAMO 2001 P4) Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

AoPS discussion thread

7. (USAMO 2003 P4) Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

AoPS discussion thread

8. (USAMO 2004 P1) Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60 degrees. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \leq |BC^3 - CD^3| \leq 3|AB^3 - AD^3|.$$

When does equality hold?

AoPS discussion thread

9. (USAMO 2010 P1) Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

AoPS discussion thread

10. (USAMO 2010 P4) Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

AoPS discussion thread

11. (USAMO 2020 P1) Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

AoPS discussion thread

3.2 10 M

12. (IMO 2000, P1) Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

   AoPS discussion thread

13. (IMO 2001, P1) Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$.

   AoPS discussion thread

14. (IMO 2002, P2) The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120^\circ$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.

   AoPS discussion thread

15. (IMO 2003, P4) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

   AoPS discussion thread

16. (IMO 2008, P1) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

   Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

   AoPS discussion thread

17. (IMO 2010, P4) Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

   AoPS discussion thread

18. (IMO 2013, P4) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

AoPS discussion thread

19. (IMO 2018, P1) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

AoPS discussion thread

20. (IMO 2020, P1) Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

AoPS discussion thread

21. (USAMO 2000 P5) Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles $\omega_2, \omega_3, \ldots, \omega_7$ such that for $k = 2, 3, \ldots, 7$, ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

AoPS discussion thread

22. (USAMO 2001 P2) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

AoPS discussion thread

23. (USAMO 2009 P1) Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and

S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

AoPS discussion thread

24. (USA TST 2014, P1) Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point. (Specifically: prove that there is a point F, determined by triangle ABC, such that no matter where X is on arc BC, line ℓ passes through F.)

AoPS discussion thread

25. (USA TST 2014, P5) Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

AoPS discussion thread

26. (USA TSTST 2017, P1) Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90°$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF. Prove that $PR \perp OH$.

AoPS discussion thread

3.3 15 M

27. (IMO 2001, P5) Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

AoPS discussion thread

28. (IMO 2007, P2) Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove

that ℓ is the bisector of angle DAB.

AoPS discussion thread

29. (IMO 2009, P2) Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

AoPS discussion thread

30. (IMO 2009, P4) Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

AoPS discussion thread

31. (IMO 2015, P4) Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

    Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

AoPS discussion thread

32. (IMO 2016, P1) Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

AoPS discussion thread

33. (IMO 2017, P4) Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R.

Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

AoPS discussion thread

34. (USAMO 2006 P6) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

AoPS discussion thread

35. (USAMO 2007 P2) A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

AoPS discussion thread

36. (USAMO 2013 P1) In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

AoPS discussion thread

37. (USAMO 2015 P2) Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

AoPS discussion thread

38. (USA TSTST 2014, P2) Consider a convex pentagon circumscribed about a circle. We name the lines that connect vertices of the pentagon with the opposite points of tangency with the circle gergonnians.

    (a) Prove that if four gergonnians are conncurrent, the all five of them are concurrent.

    (b) Prove that if there is a triple of gergonnians that are concurrent, then there is another triple of gergonnians that are concurrent.

AoPS discussion thread

39. (USA TSTST 2017, P5) Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the

line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

AoPS discussion thread

3.4 20 M

40. (IMO 2004, P5) In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

AoPS discussion thread

41. (IMO 2005, P1) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

AoPS discussion thread

42. (IMO 2010, P2) Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

AoPS discussion thread

43. (USAMO 2005 P3) Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

AoPS discussion thread

44. (USAMO 2011 P5) Let P be a given point inside quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP,$$

$$\angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

AoPS discussion thread

45. (USA TST 2015, P1) Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90^\circ$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90^\circ$ or $\angle PQF = 90^\circ$.

AoPS discussion thread

46. (USA TST 2016, P2) Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

AoPS discussion thread

47. (USA TST 2019, P1) Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

AoPS discussion thread

48. (USA TSTST 2015, P2) Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC. (The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)

AoPS discussion thread

49. (USA TSTST 2018, P5) Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

AoPS discussion thread

50. (USA TSTST 2019, P2) Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

   - Show that lines PE and QD meet on ω.
   - Prove that if ω passes through H, then lines PD and QE meet on ω as well.

AoPS discussion thread

3.5 25 M

51. (IMO 2005, P5) Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.

   Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

AoPS discussion thread

52. (IMO 2012, P5) Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

   Show that $MK = ML$.

AoPS discussion thread

53. (IMO 2015, P3) Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

   Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

AoPS discussion thread

54. (IMO 2019, P2) In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

    Prove that points P, Q, P_1, and Q_1 are concyclic.

    AoPS discussion thread

55. (USAMO 2003 P2) A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

    AoPS discussion thread

56. (USAMO 2008 P2) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

    AoPS discussion thread

57. (USAMO 2009 P5) Trapezoid $ABCD$, with $\overline{AB}||\overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersects $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

    AoPS discussion thread

58. (USAMO 2012 P5) Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

    AoPS discussion thread

59. (USAMO 2013 P6) Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

AoPS discussion thread

60. (USAMO 2014 P5) Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of the angle $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

AoPS discussion thread

61. (USAMO 2018 P5) In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

AoPS discussion thread

62. (USAMO 2019 P2) Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

AoPS discussion thread

63. (USA TST 2017, P2) Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90°$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

    - Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
    - Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

AoPS discussion thread

64. (USA TSTST 2019, P5) Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

AoPS discussion thread

3.6 30 M

65. (USA TST 2017, P5) Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

AoPS discussion thread

66. (USA TST 2018, P5) Let $ABCD$ be a convex cyclic quadrilateral which is not a kite, but whose diagonals are perpendicular and meet at H. Denote by M and N the midpoints of $\overline{BC}$ and $\overline{CD}$. Rays MH and NH meet $\overline{AD}$ and $\overline{AB}$ at S and T, respectively. Prove that there exists a point E, lying outside quadrilateral $ABCD$, such that

    - ray EH bisects both angles $\angle BES$, $\angle TED$, and
    - $\angle BEN = \angle MED$.

AoPS discussion thread

67. (USA TSTST 2016, P2) Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

AoPS discussion thread

68. (USA TSTST 2018, P3) Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

    Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60°$, prove that these four lines are concurrent on Γ.

AoPS discussion thread

3.7 35 M

69. (IMO 2000, P6) Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and

H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

AoPS discussion thread

70. (IMO 2013, P3) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

AoPS discussion thread

71. (IMO 2019, P6) Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

    Prove that lines DI and PQ meet on the line through A perpendicular to AI.

AoPS discussion thread

72. (USAMO 2007 P6) Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that
$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \leq R^3 \ ,$$

AoPS discussion thread

73. (USAMO 2017 P3) Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

AoPS discussion thread

74. (USA TST 2015, P6) Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be

a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.

AoPS discussion thread

75. (USA TST 2020, P2) Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

    Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

AoPS discussion thread

3.8 40 M

76. (IMO 2003, P3) Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

AoPS discussion thread

77. (IMO 2008, P6) Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

AoPS discussion thread

78. (IMO 2014, P3) Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

    Prove that line BD is tangent to the circumcircle of triangle TSH.

AoPS discussion thread

79. (USAMO 2011 P3) In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$,

$BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

AoPS discussion thread

80. (USA TST 2016, P6) Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90°$.

AoPS discussion thread

81. (USA TST 2019, P6) Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90°$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

   Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

AoPS discussion thread

82. (USA TSTST 2019, P9) Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.

AoPS discussion thread

3.9 45 M

83. (IMO 2002, P6) Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

AoPS discussion thread

84. (IMO 2011, P6) Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

AoPS discussion thread

85. (IMO 2018, P6) A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

AoPS discussion thread

86. (USAMO 2016 P3) Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_BF}$ and $\overleftrightarrow{I_CE}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

AoPS discussion thread

87. (USAMO 2016 P5) An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

AoPS discussion thread

88. (USA TSTST 2016, P6) Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$. AoPS discussion thread

3.10 50 M

3.11 55 M

89. (USA TST 2020, P6) Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points $Q_1, Q_2, \ldots, Q_{100}$ are concyclic.

AoPS discussion thread

4 Number Theory

4.1 0 M

1. (USAMO 2003 P1) Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

AoPS discussion thread

4.2 5 M

2. (IMO 2002, P4) Let $n \geq 2$ be a positive integer, with divisors

$$1 = d_1 < d_2 < \ \ldots < d_k = n.$$

Prove that $d_1 d_2 + d_2 d_3 + \ \ldots \ + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

AoPS discussion thread

3. (IMO 2009, P1) Let n be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

AoPS discussion thread

4. (IMO 2013, P1) Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

AoPS discussion thread

5. (IMO 2017, P1) For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

AoPS discussion thread

6. (USAMO 2006 P1) Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p-1$.

Note: For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.

AoPS discussion thread

7. (USAMO 2011 P4) Consider the assertion that for each positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

AoPS discussion thread

8. (USAMO 2014 P1) Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

AoPS discussion thread

9. (USA TSTST 2017, P4) Find all nonnegative integer solutions to
$$2^a + 3^b + 5^c = n!.$$

AoPS discussion thread

4.3 10 M

10. (IMO 2005, P4) Determine all positive integers relatively prime to all the terms of the infinite sequence
$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

AoPS discussion thread

11. (IMO 2006, P4) Determine all pairs (x, y) of integers such that
$$1 + 2^x + 2^{2x+1} = y^2.$$

AoPS discussion thread

12. (IMO 2011, P1) Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

AoPS discussion thread

13. (IMO 2016, P4) A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set
$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$
is fragrant?

AoPS discussion thread

14. (IMO 2019, P4) Find all pairs (k, n) of positive integers such that
$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

AoPS discussion thread

15. (USAMO 2005 P2) Prove that the system
$$x^6 + x^3 + x^3y + y = 147^{157}$$
$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$
has no solutions in integers x, y, and z.

AoPS discussion thread

16. (USAMO 2008 P1) Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0k_1 \ldots k_n - 1$ is the product of two consecutive integers.

AoPS discussion thread

17. (USA TST 2016, P4) Let $\sqrt{3} = 1.b_1b_2b_3\ldots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits b_n, b_{n+1}, $\ldots$, b_{2n} equals 1.

AoPS discussion thread

18. (USA TST 2019, P4) We say that a function $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is great if for any nonnegative integers m and n,
$$f(m+1, n+1)f(m, n) - f(m+1, n)f(m, n+1) = 1.$$
If $A = (a_0, a_1, \ldots)$ and $B = (b_0, b_1, \ldots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n, 0) = a_n$ and $f(0, n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

AoPS discussion thread

19. (USA TSTST 2015, P5) Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

AoPS discussion thread

20. (USA TSTST 2018, P1) As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta : \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$,

- $\theta(p + 1) = \theta(p) + 1$, and
- if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

AoPS discussion thread

21. (USA TSTST 2018, P4) For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root.

(1) Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n \in S} \frac{1}{n} \leq 1.$$

(2) Prove that there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

AoPS discussion thread

22. (USA TSTST 2019, P7) Let $f : \mathbb{Z} \to \{1, 2, \ldots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

AoPS discussion thread

4.4 15 M

23. (IMO 2011, P5) Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

AoPS discussion thread

24. (USAMO 2010 P5) Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

AoPS discussion thread

25. (USAMO 2015 P1) Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

AoPS discussion thread

26. (USAMO 2016 P2) Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

AoPS discussion thread

27. (USAMO 2019 P5) Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

AoPS discussion thread

4.5 20 M

28. (IMO 2000 Problem 5) Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

AoPS discussion thread

29. (IMO 2018, P5) Let a_1, a_2, $\ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

AoPS discussion thread

30. (IMO 2020, P5) A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

AoPS discussion thread

31. (USAMO 2004 P2) Suppose $a_1, \ldots, a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

    (a) For $i = 1, \ldots, n$, $a_i \in S$.

    (b) For $i, j = 1, \ldots, n$ (not necessarily distinct), $a_i - a_j \in S$.

    (c) For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

    Prove that S must be equal to the set of all integers.

AoPS discussion thread

32. (USA TSTST 2018, P8) For which positive integers $b > 2$ do there exist infinitely many positive integers n such that n^2 divides $b^n + 1$?

AoPS discussion thread

4.6 25 M

33. (IMO 2007, P5) Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2 - 1)^2$, then $a = b$.

AoPS discussion thread

34. (USAMO 2001 P5) Let S be a set of integers (not necessarily positive) such that

    (a) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$; (b) if x and y are elements of S (possibly equal), then $x^2 - y$ also belongs to S.

    Prove that S is the set of all integers.

AoPS discussion thread

35. (USAMO 2007 P5) Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

AoPS discussion thread

36. (USAMO 2009 P6) Let $s_1, s_2, s_3, \ldots$ be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \ldots$. Suppose that $t_1, t_2, t_3, \ldots$ is also an infinite, nonconstant sequence of rational numbers

with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j. Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j.

AoPS discussion thread

37. (USA TST 2015, P2) Prove that for every $n \in \mathbb{N}$, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

AoPS discussion thread

38. (USA TST 2018, P1) Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

AoPS discussion thread

39. (USA TST 2020, P5) Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:
    - $m > 1$ and $\gcd(m, n) = 1$;
    - the numbers $P(0)$, $P^2(0)$, …, $P^{m-1}(0)$ are not divisible by n; and
    - $P^m(0)$ is divisible by n. Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

AoPS discussion thread

40. (USA TSTST 2014, P6) Suppose we have distinct positive integers a, b, c, d, and an odd prime p not dividing any of them, and an integer M such that if one considers the infinite sequence

$$\begin{gathered} ca - db \\ ca^2 - db^2 \\ ca^3 - db^3 \\ ca^4 - db^4 \\ \vdots \end{gathered}$$

and looks at the highest power of p that divides each of them, these powers are not all zero, and are all at most M.

Prove that there exists some T (which may depend on a, b, c, d, p, M) such that whenever p divides an element of this sequence, the maximum power of p that divides that element is exactly p^T. AoPS discussion thread

4.7 30 M

41. (IMO 2006, P5) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\dots P(P(x))\dots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

AoPS discussion thread

42. (IMO 2008, P3) Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$

AoPS discussion thread

43. (IMO 2015, P2) Find all positive integers (a, b, c) such that

$$ab - c, \quad bc - a, \quad ca - b$$

are all powers of 2.

AoPS discussion thread

44. (USAMO 2013 P5) Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

AoPS discussion thread

45. (USAMO 2015 P5) Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

AoPS discussion thread

46. (USA TST 2014, P6) For a prime p, a subset S of residues modulo p is called a sum-free multiplicative subgroup of $\mathbb{F}_p$ if
    - there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \dots\}$ (all considered mod p), and
    - there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod{p}$.

    Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

AoPS discussion thread

47. (USA TST 2016, P3) Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$.

Define $\Psi : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

Here, a polynomial Q divides P if there exists $R \in \mathbb{F}_p[x]$ such that $P(x) - Q(x)R(x)$ is the polynomial with all coefficients 0 (with all addition and multiplication in the coefficients taken modulo p), and the gcd of two polynomials is the highest degree polynomial with leading coefficient 1 which divides both of them. A non-zero polynomial is a polynomial with not all coefficients 0. As an example of multiplication, $(x+1)(x+2)(x+3) = x^3 + x^2 + x + 1$ in $\mathbb{F}_5[x]$.

AoPS discussion thread

4.8 35 M

48. (IMO 2003, P2) Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

AoPS discussion thread

49. (IMO 2003, P6) Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

AoPS discussion thread

50. (IMO 2004, P6) We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

AoPS discussion thread

51. (USAMO 2005 P6) For m a positive integer, let $s(m)$ be the sum of the digits of m. For $n \geq 2$, let $f(n)$ be the minimal k for which there exists a set S of n positive integers such that $s\left(\sum_{x \in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

AoPS discussion thread

52. (USAMO 2006 P3) For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence

$$\{p\left(f\left(n^2\right)\right) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires $f\left(n^2\right) \neq 0$ for $n \geq 0$.)

AoPS discussion thread

53. (USAMO 2012 P3) Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \ldots + na_{nk} = 0$$

holds for every positive integer k.

AoPS discussion thread

54. (USA TST 2017, P6) Prove that there are infinitely many triples (a, b, p) of positive integers with p prime, $a < p$, and $b < p$, such that $(a+b)^p - a^p - b^p$ is a multiple of p^3.

AoPS discussion thread

4.9 40 M

55. (IMO 2016, P3) Let $P = A_1 A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n

AoPS discussion thread

56. (IMO 2017, P6) An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n = 1.$$

AoPS discussion thread

57. (USAMO 2014 P6) Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all

$i, j \in \{0, 1, \dots n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

AoPS discussion thread

58. (USAMO 2019 P3) Let K be the set of all positive integers that do not contain the digit 7 in their base-10 representation. Find all polynomials f with nonnegative integer coefficients such that $f(n) \in K$ whenever $n \in K$.

AoPS discussion thread

59. (USAMO 2020 P3) Let p be an odd prime. An integer x is called a quadratic non-residue if p does not divide $x - t^2$ for any integer t.

    Denote by A the set of all integers a such that $1 \le a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

AoPS discussion thread

60. (USA TST 2014, P2) Let $a_1, a_2, a_3, \dots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

    is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

AoPS discussion thread

61. (USA TST 2019, P2) Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \dots, \quad g(x) + 100x$$

    are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

AoPS discussion thread

62. (USA TSTST 2015, P3) Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $p_1, p_2, ..., p_k$ of M, all prime factors of $p_1p_2...p_k + 1$ are also in M. Prove that $M = P$.

AoPS discussion thread

63. (USA TSTST 2016, P3) Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for

every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

AoPS discussion thread

4.10 45 M

64. (USAMO 2018 P3) For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.

AoPS discussion thread

4.11 50 M

65. (USA TSTST 2019, P6) Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant? (A Fibonacci number is an element of the sequence $F_0, F_1, \ldots$ defined recursively by $F_0 = 0, F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.)

AoPS discussion thread