A Beautiful Journey
Through Olympiad Geometry

Stefan Lozanovski

A Beautiful Journey Through Olympiad Geometry

Stefan Lozanovski

Version 1.4
www.olympiadgeometry.com

Cover created by Damjan Lozanovski

All illustrations created with GeoGebra (www.geogebra.org)

Contents

Introduction

This book is aimed at anyone who wishes to prepare for the geometry part of the mathematics competitions and Olympiads around the world. No previous knowledge of geometry is needed. Even though I am a fan of non-linear storytelling, this book progresses in a linear way, so everything that you need to know at a certain point will have been already visited before. We will start our journey with the most basic topics and gradually progress towards the more advanced ones. The level ranges from junior competitions in your local area, through senior national Olympiads around the world, to the most prestigious International Mathematical Olympiad.

The word “Beautiful” in the book’s title means that we will explore only synthetic approaches and proofs, which I find elegant and beautiful. We will not see any analytic approaches, such as Cartesian or barycentric coordinates, nor we will do complex number or trigonometry bashing.

Structure

This book is structured in two parts. The first one provides an introduction to concepts and theorems. For the purpose of applying these concepts and theorems to geometry problems, a number of useful properties and examples with solutions are offered. At the end of each chapter, a selection of unsolved problems is provided as an exercise and a challenge for the reader to test their skills in relation to the chapter topics. This part can be roughly divided in two portions: Junior (the first 11 chapters) and Senior (the other 16 chapters). The second part of this book contains mixed problems, mostly from competitions and Olympiads from all around the world.

Acknowledgments

I would like to thank my primary school math teacher Ms. Vesna Todorovikj for her dedication in training me and my friend Bojan Joveski for the national math competitions. She introduced me to problem solving and thinking logically, in general. I’ll never forget the handwritten collection of geometry problems that she gave us, which made me start loving geometry.

I would also like to thank my high school math Olympiad mentor, Mr. Özgür Kırçak. He boosted my Olympiad spirit during the many Saturdays in “Olympiad Room” while eating burek, drinking tea and solving Olympiad problems. Under his guidance, I started preparing geometry worksheets and teaching the younger Olympiad students. Those worksheets are the foundation of this book.

Finally, I would like to thank all of my students for working through the geometry worksheets, shaping the Olympiad geometry curriculum together with me and giving honest feedback about the lessons and about me as a teacher. Their enthusiasm for geometry and thirst for more knowledge were a great inspiration for me to write this book. I would especially like to thank Nikola Danevski, who helped me add some important topics in version 1.3 of this book, specifically by writing the sections about $\sqrt{bc}$ Inversion and co-writing the chapter Spiral Similarity.

Support & Feedback

This book is part of my project for sharing knowledge with the whole world. If you are satisfied with the book contents, please support the project by donating at olympiadgeometry.com.

Tell me what you think about the book and help me make this Journey even more beautiful. Write a general comment about the book, suggest a topic you'd like to see covered in a future version or report a mistake at the same web site.

You can also follow us on Facebook (facebook.com/olympiadgeometry) for the latest news and updates. Please leave you honest review there.

The Author

Notations

Since the math notations slightly differ in various regions of the world, here is a quick summary of the ones we are going to use throughout our journey.

Notation	Explanation
$\angle ABC$	angle ABC; or measurement of said angle
$\overline{AB}$	length of the line segment AB
$\overrightarrow{AB}$	vector AB
$\widehat{AB}$	arc AB
$(ABCD)$	circumcircle of the cyclic polygon $ABCD$
$\equiv$	coincide Example: if $A - B - C$ are collinear, then $AB \equiv AC$.
$\cap$	intersection
$\perp$	perpendicular
$\parallel$	parallel
$P_{\triangle ABC}$	area of the triangle ABC
P_{ABCD}	area of the polygon $ABCD$
$d(P,\, AB)$	distance from the point P to the line AB
$\angle(p, q)$	angle between the lines p and q
$\measuredangle(p, q)$	directed angle between the lines p and q
$\alpha, \beta, \gamma, \dots$	unless otherwise noted, the angles at the vertices $A, B, C, \dots$ in a polygon $ABC\dots$; or measurements of said angles
a, b, c	unless otherwise noted, the sides opposite the vertices A, B, C in a triangle ABC; or lengths of said sides
$\iff$	if and only if (shortened iff) Example: $p \iff q$ means "if p then q AND if q then p".
$\therefore$	therefore
$\because$	because
LHS \ RHS	The left-hand side \ the right-hand side of an equation
WLOG	Without loss of generality
w.r.t.	with respect to
$\blacksquare$	Q.E.D. (initialism of the Latin phrase "quod erat demonstrandum", meaning "which is what had to be proven".)

Part I

Lessons

Chapter 1

Congruence of Triangles

Two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are said to be congruent when their corresponding sides and corresponding angles are equal.

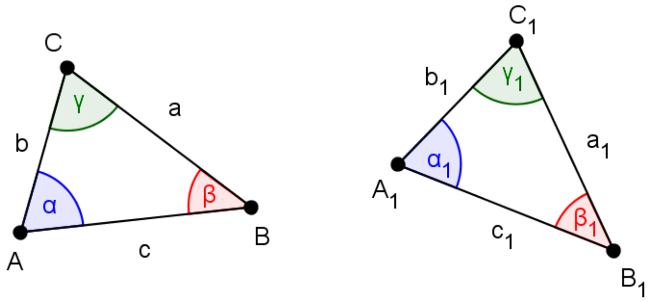


Figure 1.1: Congruent triangles.

$$\triangle ABC \cong \triangle A_1B_1C_1 \iff a = a_1, b = b_1, c = c_1, \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1$$

However, in most of the problems, the equality of all these six pairs of elements will not be given, so we will need to use some criteria for congruence. With these criteria, we will prove the congruence of two triangles only by using the equality of three pairs of corresponding elements.

Criterion SSS (side-side-side) If three pairs of corresponding sides are equal, then the triangles are congruent.

Criterion SAS (side-angle-side) If two pairs of corresponding sides and the angles between them are equal, then the triangles are congruent.

Criterion ASA (angle-side-angle) If two pairs of corresponding angles and the sides formed by the common rays of these angles are equal, then the triangles are congruent.

These criteria are part of our axioms, so we will not prove them. However, in Figure 1.2, you can see that we can construct exactly one triangle given the corresponding set of elements for each criteria. We can also see why there can not exist an ASS congruence criterion.

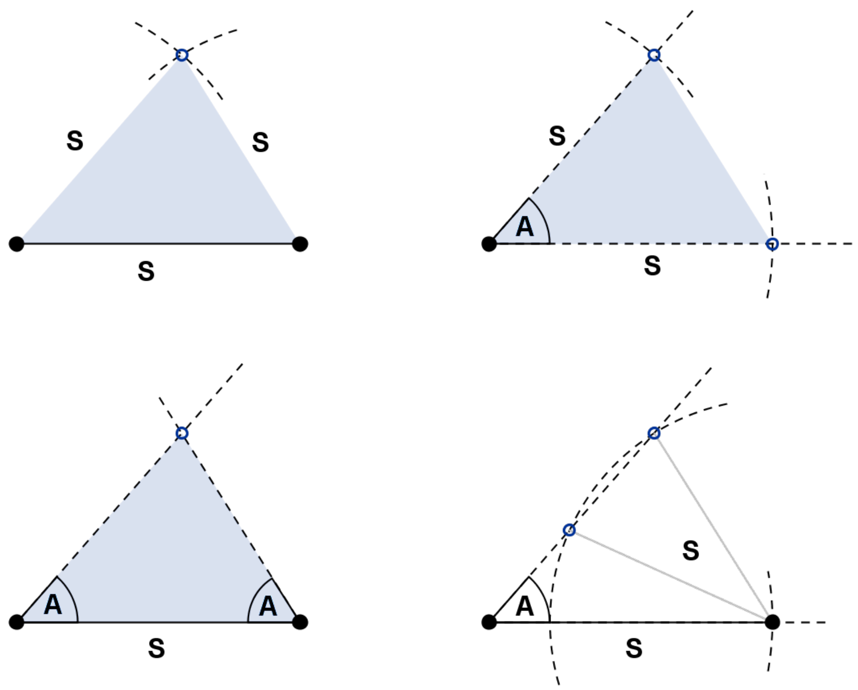


Figure 1.2: Criteria for congruence of triangles.

Chapter 2

Angles of a Transversal

When two lines p and q are intersected by a third line t, we get eight angles. The line t is called a *transversal*. The pairs of angles, depending on their position relative to the transversal and the two given lines are called:

corresponding angles if they lie on the same side of the transversal and one of them is in the interior of the lines p and q, while the other one is in the exterior (e.g. α_1 and α_2);

alternate angles if they lie on different side of the transversal and both of them are either in the interior or in the exterior of the lines p and q (e.g. β_1 and β_2); or

opposite[1]**angles** if they lie on the same side of the transversal and both of them are either in the interior or in the exterior of the lines p and q (e.g. γ_1 and γ_2).

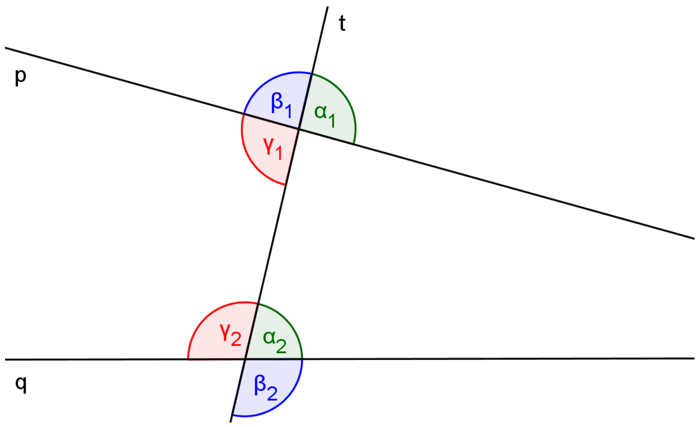


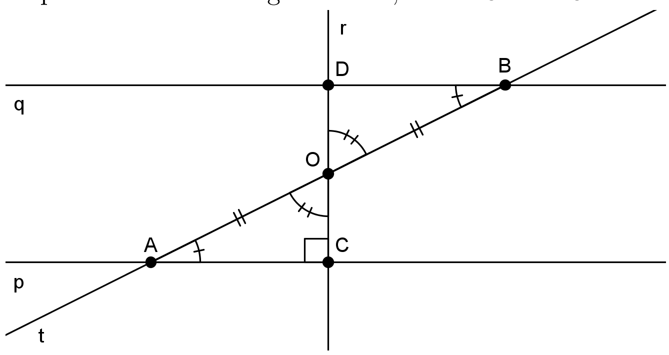
Figure 2.1: Angles of a transversal.

[1]In some resources, the interior opposite angles are called *consecutive interior angles*, but there is no name for the exterior opposite angles, which have the same property. Since in some languages these angles are called opposite, in this book we'll call them that in English, too, even though I haven't seen this terminology used in other resources in English.

Property 2.1. If the lines p and q are parallel, then the corresponding angles are equal, the alternate angles are equal and the opposite angles are supplementary. The converse is also true.

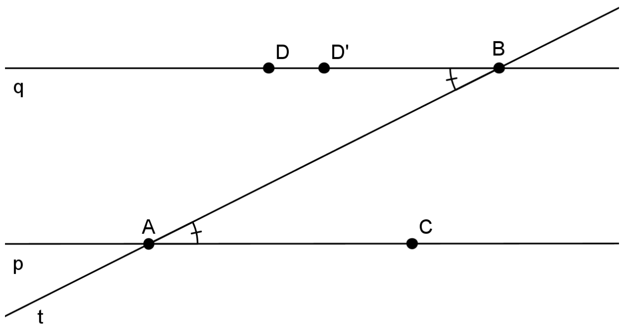
$$p \parallel q \iff \alpha_1 = \alpha_2,\ \beta_1 = \beta_2,\ \gamma_1 + \gamma_2 = 180^\circ \tag{2.1}$$

Proof. Let the transversal t intersect p and q at A and B, respectively and let O be the midpoint of the line segment AB, i.e. $\overline{AO} = \overline{BO}$. Let r be a line



through O that is perpendicular to p. Let $r \cap p = C$ and $r \cap q = D$. Then $\angle OCA = 90^\circ$. Let's prove one of the directions, i.e. let $\angle OAC = \angle OBD$. The angles $\angle AOC$ and $\angle BOD$ are vertical angles and therefore equal. So, by the criterion ASA, $\triangle AOC \cong \triangle BOD$. Therefore, their corresponding elements are equal, i.e. $\angle ODB = \angle OCA = 90^\circ$. So, $r \perp q$. Therefore, $p \parallel q$. □

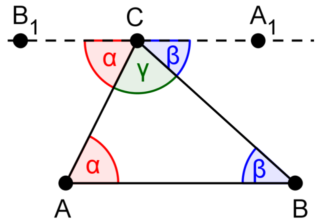
Now, let's prove the other direction. Let $p \parallel q$. Let t be a transversal, such



that $t \cap p = A$ and $t \cap q = B$. Let $C \in p$ and $D \in q$, such that C and D are on different sides of t. We want to prove that $\angle BAC = \angle ABD$. Let D' be a point such that $\angle BAC = \angle ABD'$. By the direction we just proved, $AC \parallel BD'$. Since B lies on both BD and BD' and $BD' \parallel AC \parallel BD$, then $BD \equiv BD'$ and consequently, $\angle ABD \equiv \angle ABD'$. Therefore, $\angle BAC = \angle ABD$.

Remark. The other angles with vertices at A and B are either vertical to (and therefore equal) or form a linear pair (and therefore supplementary) with the angles $\angle BAC$ and $\angle ABD$, so it is easy to prove the rest. ■

Property 2.2 (Sum of angles in triangle)**.** The sum of the interior angles in a triangle is 180 degrees.

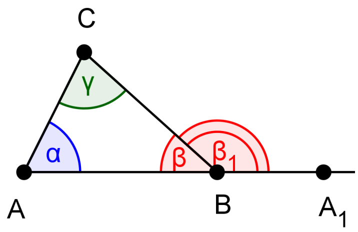


Proof. Let ABC be a triangle. Let's draw a line B_1A_1 which passes through C and is parallel to AB. Then, by Property 2.1, we have:

$$
\begin{aligned}
&\angle B_1CA = \angle CAB = \alpha \quad \text{(alternate interior angles; transversal } AC\text{)}\\
&\angle A_1CB = \angle CBA = \beta \quad \text{(alternate interior angles; transversal } BC\text{)}\\
&\angle ACB = \gamma\\
\therefore\ &\angle B_1CA + \angle A_1CB + \angle ACB = \alpha + \beta + \gamma\\
&\angle B_1CA_1 = \alpha + \beta + \gamma\\
&180^\circ = \alpha + \beta + \gamma
\end{aligned}
$$

■

Property 2.3 (Exterior angle in triangle)**.** An exterior angle in a triangle equals the sum of the two non-adjacent interior angles.



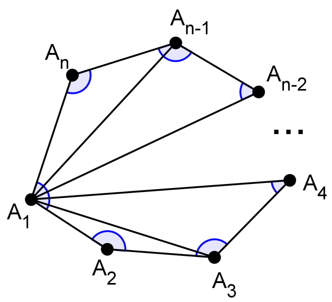
Proof. Let ABC be a triangle and let A_1 be a point on the extension of AB.

$$
\begin{aligned}
&\angle A_1BC + \angle ABC = 180^\circ \quad \text{(linear pair)}\\
&\angle ABC + \angle BCA + \angle CAB = 180^\circ \quad \text{(Sum of angles in triangle)}\\
\therefore\ &\angle A_1BC = 180^\circ - \angle ABC = \angle BCA + \angle CAB
\end{aligned}
$$

■

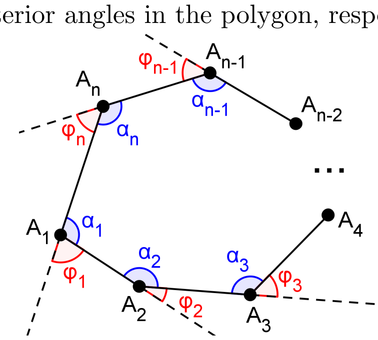
Property 2.4. Find the sum of the interior angles in an n-gon.

Proof. Let $A_1A_2A_3\dots A_n$ be a polygon with n sides. If we draw the diagonals from A_1 to all the other $(n-3)$ vertices, we get $(n-2)$ distinct triangles. By Property 2.2, the sum of all the interior angles in these triangles is $(n-2)\cdot 180^\circ$. Note that these angles actually form all the interior angles in the n-gon. So, the sum of the interior angles in an n-gon is $(n-2)\cdot 180^\circ$. ■



Property 2.5. Find the sum of the exterior angles in an n-gon.

Proof. Let $A_1A_2A_3\dots A_n$ be a polygon with n sides. Let α_i and φ_i $(i = 1, 2, \dots, n)$ be the interior and exterior angles in the polygon, respectively.



Since each exterior and its corresponding interior angle form a linear pair, we have $\alpha_i + \beta_i = 180°,\ i = 1, 2, \dots, n$. If we sum these equations, we get

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \varphi_i = n \cdot 180°.$$

From Property 2.4, we know that

$$\sum_{i=1}^{n} \alpha_i = (n-2) \cdot 180°.$$

In order to find the sum of the exterior angles, we need to subtract the two previous equations.

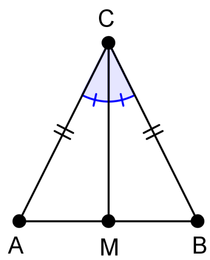
$$\sum_{i=1}^{n} \varphi_i = \big(n - (n-2)\big) \cdot 180° = 2 \cdot 180° = 360°.$$

So, the sum of the exterior angles in any polygon does not depend on the number of sides n and is always $360°$. ■

Property 2.6 (Isosceles Triangle)**.** In $\triangle ABC$, if two of the sides are equal, then the corresponding angles are equal, i.e. if $\overline{CA} = \overline{CB}$, then $\angle CAB = \angle CBA$.

Proof. Let the angle bisector of $\angle BCA$ intersect the side AB at M. Then, $\angle ACM = \angle BCM$. Combining with $\overline{CA} = \overline{CB}$ and CM-common side, by SAS, we get that $\triangle ACM \cong \triangle BCM$. Therefore, their corresponding angles are equal, i.e.

$$\angle CAB \equiv \angle CAM = \angle CBM \equiv \angle CBA.$$



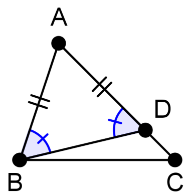
Additionally, as a consequence of the congruence, we can also get two other things: $\overline{AM} = \overline{MB}$ and $\angle AMC = \angle BMC$, which means that $CM \perp AB$. Therefore, as a conclusion, the angle bisector, the median and the altitude from the vertex C in an isosceles triangle coincide with the side bisector of AB. ■

Remark. The converse is also true (if $\angle CAB = \angle CBA$, then $\overline{CA} = \overline{CB}$). Can you prove it by yourself?

Property 2.7 (Equilateral triangle)**.** In $\triangle ABC$, all three sides are equal. Prove that all the angles are equal to $60°$.

Proof. Combining Property 2.2 and Property 2.6, we directly get the desired result. ■

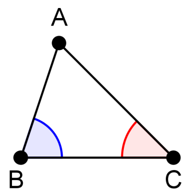
Property 2.8. In any triangle, a greater side subtends a greater angle.



Proof. In $\triangle ABC$, let $\overline{AC} > \overline{AB}$. Then we can choose a point D on the side AC, such that $\overline{AD} = \overline{AB}$. Since $\triangle ABD$ is isosceles, we have $\angle ABD = \angle ADB$.

$$\angle ABC > \angle ABD = \angle ADB \overset{2.3}{=} \angle DBC + \angle DCB > \angle DCB \equiv \angle ACB$$ ■

Property 2.9. In any triangle, a greater angle is subtended by a greater side.



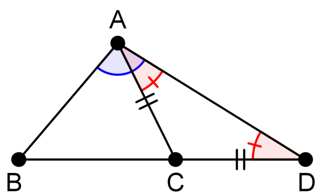
Proof. In $\triangle ABC$, let $\angle ABC > \angle ACB$. We want to prove that $\overline{AC} > \overline{AB}$. Let's assume the opposite, i.e. $\overline{AC} \leq \overline{AB}$.

i) If $\overline{AC} = \overline{AB}$, then by Property 2.6, $\angle ABC = \angle ACB$, which is not true.

ii) If $\overline{AC} < \overline{AB}$, then by Property 2.8, $\angle ABC < \angle ACB$, which is not true.

Therefore, our assumption is wrong, so $\overline{AC} > \overline{AB}$. ■

Property 2.10 (Triangle Inequality)**.** In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.



Proof. In $\triangle ABC$, let D be a point on the extension of the side BC beyond C, such that $\overline{CD} = \overline{CA}$. Then, $\triangle CAD$ is isosceles, so $\angle CAD = \angle CDA$. Now, in $\triangle BAD$ we have

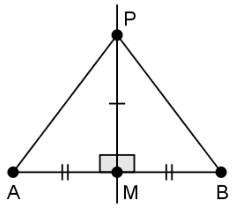
$$\angle BAD = \angle BAC + \angle CAD > \angle CAD = \angle CDA \equiv \angle BDA,$$

which by Property 2.9 means that $\overline{BD} > \overline{AB}$. Therefore,

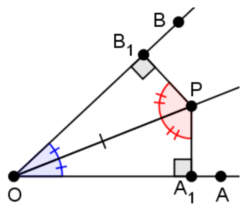
$$\overline{BC} + \overline{CA} = \overline{BC} + \overline{CD} = \overline{BD} > \overline{AB}$$ ■

Property 2.11. Any point P that lies on the side bisector of a line segment AB is equidistant from the endpoints.

Proof. Let p and M be the side bisector and the midpoint of AB, respectively. Therefore, $M \in p$. As $p \perp AB$, we have $\angle PMA = \angle PMB = 90°$. Combining with $\overline{MA} = \overline{MB}$ and MP - common side, we get $\triangle PMA \cong \triangle PMB$ (by the SAS criterion). Therefore, $\overline{PA} = \overline{PB}$, i.e. P is equidistant from the endpoints. ■



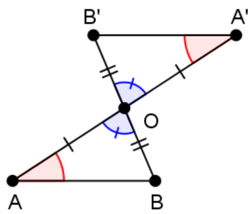
Property 2.12. Any point P that lies on the angle bisector of an angle $\angle AOB$ is equidistant from the rays.



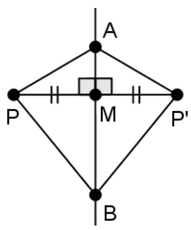
Proof. Let p be the angle bisector of $\angle AOB$ and let A_1, B_1 be the feet of the perpendiculars from P to OA, OB, respectively. Therefore, $\angle POA_1 = \angle POB_1 = \frac{\alpha}{2}$ and $\angle OPA_1 = 90° - \frac{\alpha}{2} = \angle OPB_1$. Since OP is a common side, we get $\triangle OPA_1 \cong \triangle OPB_1$ (by the ASA criterion). Therefore, $\overline{PA_1} = \overline{PB_1}$, i.e. P is equidistant from the rays of $\angle AOB$. ■

Property 2.13. Let A' and B' be the reflections of the points A and B, respectively, with respect to the point O. Prove that $\overline{AB} = \overline{A'B'}$ and $AB \parallel A'B'$.

Proof. Because we have $\overline{OA} = \overline{OA'}$, $\overline{OB} = \overline{OB'}$ and $\angle AOB = \angle A'OB'$ as vertical angles, by the SAS criterion we have that $\triangle OAB \cong \triangle OA'B'$. Therefore, $\overline{AB} = \overline{A'B'}$ and $\angle OAB = \angle OA'B'$ which implies that $AB \parallel A'B'$ because the alternate angles of the transversal AA' and the lines AB and $A'B'$ are equal. ■



Property 2.14. Let P' be the reflection of the point P with respect to the line AB. Prove that $\triangle PAB \cong \triangle P'AB$.



Proof. Let the intersection of AB and PP' be M. Then, $\overline{PM} = \overline{MP'}$ and $PM \perp AB$. Since $\overline{PM} = \overline{MP'}$, $\angle PMA = 90° = \angle P'MA$ and AM is a common side, by the SAS criterion we get that $\triangle PMA \cong \triangle P'MA$ and therefore $\overline{PA} = \overline{P'A}$. Similarly, $\triangle PMB \cong \triangle P'MB$ and therefore $\overline{PB} = \overline{P'B}$. Finally, by the SSS criterion we get that $\triangle PAB \cong \triangle P'AB$. ■

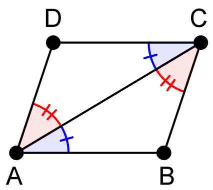
The quadrilaterals, depending on the number of parallel opposite sides, are divided in 2 categories:

trapezoid [2] with at least 1 pair of parallel opposite sides

parallelogram with 2 pairs of parallel opposite sides

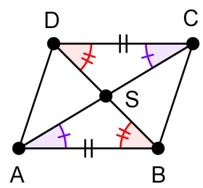
Now, we will present 2 properties of the parallelograms and later we will present 3 criteria for parallelograms, i.e. we will show 3 different ways how to prove that a quadrilateral is a parallelogram (apart from the obvious way, by definition, i.e. by proving that both pairs of opposite sides are parallel).

Property 2.15. Let $ABCD$ be a parallelogram. Prove that its opposite sides are of equal length.



Proof. Let's draw the diagonal AC. Since $AB \parallel CD$, by Property 2.1, $\angle CAB = \angle ACD$. Similarly, since $BC \parallel AD$, $\angle ACB = \angle CAD$. Therefore, since AC is a common side for the triangles $\triangle ABC$ and $\triangle CDA$, by the ASA criterion, $\triangle ABC \cong \triangle CDA$. Therefore, their corresponding elements, are equal, i.e. $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DA}$. ■

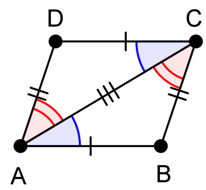
Property 2.16. Let $ABCD$ be a parallelogram. Prove that its diagonals bisect at their intersection point.



Proof. Let the intersection of the diagonals AC and BD be S. Because $AB \parallel CD$, from Property 2.1 we get that $\angle SAB = \angle SCD$. Similarly, $\angle SBA = \angle SDC$. Also, from Property 2.15 we know that $\overline{AB} = \overline{CD}$, so by combining these three facts, by the ASA criterion we get that $\triangle SAB \cong \triangle SCD$. Therefore, $\overline{SA} = \overline{SC}$ and $\overline{SB} = \overline{SD}$, i.e. the diagonals bisect at their intersection point. ■

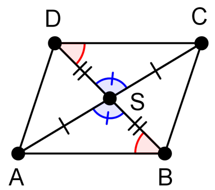
[2] American English (in British English, a quadrilateral with 1 pair of parallel opposite sides is called a "trapezium", while the term "trapezoid" refers to a quadrilateral with no parallel opposite sides). In this book, we will use the American English terminology.

Property 2.17. In the quadrilateral $ABCD$, the opposite sides are of equal length. Prove that $ABCD$ is a parallelogram.



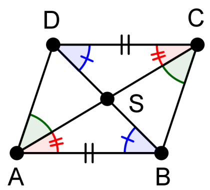
Proof. Let's draw the diagonal AC. Since $\overline{AB} = \overline{CD}$, $\overline{BC} = \overline{DA}$ and AC is a common side, by the SSS criterion we get that $\triangle ABC \cong \triangle CDA$. Therefore $\angle BAC = \angle DCA$, which by Property 2.1 implies that $AB \parallel CD$. Similarly, $\angle BCA = \angle DAC$ and therefore $BC \parallel AD$. Hence, $ABCD$ is a parallelogram. ■

Property 2.18. In the quadrilateral $ABCD$, the intersection point of the diagonals bisects them. Prove that $ABCD$ is a parallelogram.



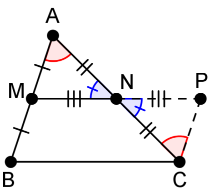
Proof. Let the intersection of the diagonals AC and BD be S. Then, from the condition, we have that $\overline{AS} = \overline{SC}$ and $\overline{BS} = \overline{SD}$. Let's take a look at $\triangle ABS$ and $\triangle CDS$. We have $\overline{AS} = \overline{CS}$, $\angle ASB = \angle CSD$ as vertical angles and $\overline{BS} = \overline{DS}$. So, by the SAS criterion, $\triangle ABS \cong \triangle CDS$. Therefore, the corresponding elements are equal, i.e. $\angle ABS = \angle CDS$. Since these angles are alternate angles of the transversal BD and the lines AB and CD, we have that $AB \parallel CD$. Similarly, $\triangle BCS \cong \triangle DAS$ and $\angle BCS = \angle DAS$. Therefore, $BC \parallel DA$. ■

Property 2.19. In the quadrilateral $ABCD$, $\overline{AB} = \overline{CD}$ and $AB \parallel CD$. Prove that $ABCD$ is a parallelogram.



Proof. Let the intersection of the diagonals AC and BD be S. Since $AB \parallel CD$, the alternate angles of the transversal BD are equal, i.e. $\angle ABS = \angle CDS$. Similarly, $\angle BAS = \angle DCS$. Combining with the fact that $\overline{AB} = \overline{CD}$, by the ASA criterion, we get that $\triangle ABS \cong \triangle CDS$. Therefore, as the corresponding elements are equal, $\overline{AS} = \overline{CS}$ and $\overline{BS} = \overline{DS}$. Combining with the fact that $\angle ASD = \angle CSB$ as vertical angles, by the SAS criterion we get that $\triangle ASD \cong \triangle CSB$. Therefore, $\angle DAS = \angle BCS$, so $DA \parallel BC$. ■

Property 2.20 (Midsegment Theorem)**.** In a triangle, the segment joining the midpoints of any two sides is parallel to the third side and half its length.



Proof. In $\triangle ABC$, let M and N be the midpoints of the sides AB and AC, respectively. Let P be a point on the ray MN beyond N, such that $\overline{MN} = \overline{NP}$. Since $\angle MNA = \angle PNC$ as vertical angles, by SAS we get $\triangle AMN \cong \triangle CPN$. Therefore, $\overline{AM} = \overline{CP}$ and $\angle MAN = \angle PCN$ which means that $AM \parallel CP$. Now, we have $\overline{BM} = \overline{AM} = \overline{CP}$ and $BM \equiv AM \parallel CP$. By Property 2.19, since the opposite sides in the quadrilateral $MBCP$ are of equal length and parallel, it must be a parallelogram. Therefore,

$$MN \equiv MP \parallel BC$$

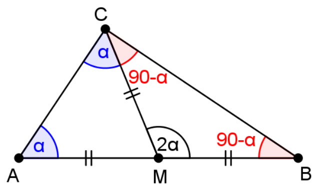
and because of Property 2.15,

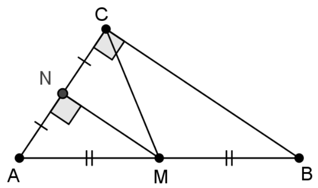
$$\overline{MN} = \frac{1}{2}\overline{MP} = \frac{1}{2}\overline{BC}. \qquad \blacksquare$$

Property 2.21. Let M be the midpoint of the side AB in the triangle ABC. Prove that $\angle ACB = 90^\circ$ if and only if $\overline{MA} = \overline{MB} = \overline{MC}$.

Proof. Let $\overline{MA} = \overline{MB} = \overline{MC}$.
Let $\angle BAC = \alpha$. Since $\triangle MAC$ is isosceles, $\angle MCA = \angle MAC \equiv \angle BAC = \alpha$. As an exterior angle of $\triangle MAC$, $\angle BMC = \angle MAC + \angle MCA = 2\alpha$. Now, since $\triangle MBC$ is isosceles, $\angle MCB = \frac{1}{2} \cdot (180^\circ - \angle BMC) = 90^\circ - \alpha$. Finally, $\angle ACB = \angle ACM + \angle MCB = 90^\circ$. $\square$

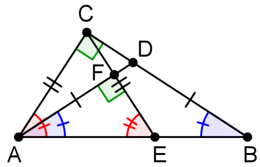
Now, let's prove the other direction. Let $\angle ACB = 90^\circ$.
Let N be the midpoint of AC. Then, MN is a midsegment in $\triangle ABC$ and therefore $MN \parallel BC$. Since $AC \perp BC$, we get $AC \perp MN$, i.e. MN is altitude in $\triangle MAC$. Since MN is both median and altitude in $\triangle MAC$, then $\triangle MAC$ is isosceles. Therefore, $\overline{MA} = \overline{MC}$. Since M is the midpoint of AB, we get $\overline{MA} = \overline{MB} = \overline{MC}$. $\blacksquare$





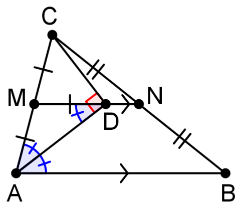
We will now solve a few examples using the things we learned in the first two chapters.

Example 2.1. Let ABC be a right triangle with $\angle BCA = 90^\circ$ and $\overline{CA} < \overline{CB}$. Let $D \in BC$, such that $\overline{DA} = \overline{DB}$ and $E \in AB$ such that $\overline{CA} = \overline{CE}$. Prove that $AD \perp CE$.



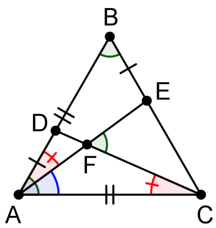
Proof. Since $\overline{CA} = \overline{CE}$, by Property 2.6, we have $\angle CEA = \angle CAE = \alpha$. Similarly, since $\overline{DA} = \overline{DB}$, we have $\angle DAB = \angle DBA = \beta$. From Sum of angles in triangle $\triangle ABC$, we get $\alpha + \beta = 90^\circ$. Now, let $AD \cap CE = F$. From $\triangle AFE$ we get $\angle AFE = 180^\circ - (\angle FEA + \angle EAF) = 180^\circ - (\alpha + \beta) = 90^\circ$. ■

Example 2.2. Let M and N be midpoints of the sides CA and CB, respectively, in a triangle ABC. The angle bisector of $\angle BAC$ intersects the line MN at D. Prove that $\angle ADC = 90^\circ$.



Proof. We have that MN is a midsegment in $\triangle ABC$, so, by Property 2.20, $MN \parallel AB$. By Property 2.1, for the transversal AD we get $\angle MDA = \angle DAB$. But, since AD is angle bisector of $\angle CAB$, we have $\angle DAB = \angle DAC$. Thus, $\angle MDA = \angle DAB = \angle DAC \equiv \angle DAM$, so $\triangle MAD$ is an Isosceles Triangle and therefore $\overline{MD} = \overline{MA}$. Since $\overline{MD} = \overline{MA} = \overline{MC}$, by Property 2.21 we get $\angle ADC = 90^\circ$. ■

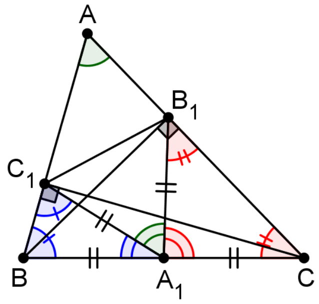
Example 2.3. Let ABC be an equilateral triangle. Let $D \in AB$ and $E \in BC$, such that $\overline{AD} = \overline{BE}$. Let $AE \cap CD = F$. Find $\angle CFE$.



Proof. We focus on the triangles $\triangle ABE$ and $\triangle CAD$. Since $\triangle ABC$ is Equilateral triangle, we get $\overline{AB} = \overline{CA}$ and $\angle ABE = 60^\circ = \angle CAD$. Using $\overline{BE} = \overline{AD}$, by the criterion SAS, we get $\triangle ABE \cong \triangle CAD$. Therefore, $\angle EAB = \angle DCA$. Finally, as an Exterior angle in triangle $\triangle FAC$, we get:

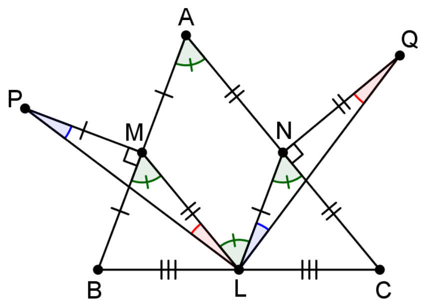
$$\begin{aligned}\angle CFE =& \angle CAF + \angle FCA \equiv \\ \equiv& \angle CAE + \angle DCA = \angle CAE + \angle EAB = \angle CAB = 60^\circ \quad \blacksquare\end{aligned}$$

Example 2.4. In the triangle ABC, let A_1 be the midpoint of BC and let B_1 and C_1 be the feet of the altitudes from the vertices B and C, respectively. Prove that the triangle $A_1B_1C_1$ is equilateral if and only if $\angle BAC = 60°$.



Proof. Since A_1 is midpoint of the hypotenuse BC in the right triangle $\triangle B_1BC$, by Property 2.21, we get $\overline{A_1B_1} = \overline{A_1B} = \overline{A_1C}$. Similarly, from $\triangle C_1BC$ we get $\overline{A_1C_1} = \overline{A_1B} = \overline{A_1C}$. Therefore, triangles $\triangle A_1B_1C_1$, $\triangle A_1BC_1$ and $\triangle A_1CB_1$ are isosceles. Therefore, $\angle BA_1C_1 = 180 - 2\beta$ and $\angle CA_1B_1 = 180° - 2\gamma$. So, $\angle B_1A_1C_1 = 180° - (\angle BA_1C_1 + \angle CA_1B_1) = 180° - 2\cdot(180° - \beta - \gamma) = 180 - 2\alpha$. Finally, we see that the isosceles $\triangle A_1B_1C_1$ is equilateral if and only if $\angle B_1A_1C_1 = 180° - 2\alpha = 60°$, which is true if and only if $\alpha = 60°$. ∎

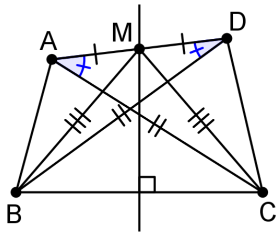
Example 2.5. Let M and N be midpoints of the sides AB and AC, respectively, in a triangle ABC. Let P and Q be points outside the triangle, such that $PM \perp AB, \overline{PM} = \frac{1}{2}\overline{AB}$ and $QN \perp AC, \overline{QN} = \frac{1}{2}\overline{AC}$. If L is the midpoint of BC, prove that $\overline{LP} = \overline{LQ}$ and $\angle PLQ = 90°$.



Proof. Since LM is a midsegment in $\triangle ABC$, we get $\overline{LM} = \frac{1}{2}\overline{CA}$ and $LM \parallel CA$. Therefore, $\overline{LM} = \overline{CN} = \overline{AN} = \overline{QN}$ and by Property 2.1, $\angle LMB = \angle CAB$, i.e. $\angle LMP = 90° + \alpha$. Similarly, $\overline{NL} = \overline{MP}$ and $\angle QNL = 90° + \alpha$. Therefore, by SAS, $\triangle LMP \cong \triangle QNL$, so $\overline{LP} = \overline{QL}$, $\angle MPL = \angle NLQ = x$ and $\angle PLM = \angle LQN = y$. Since $LM \parallel NA$ and $LN \parallel MA$, we get that $LMAN$ is a parallelogram, so $\angle MLN = \alpha$. From Sum of angles in triangle $\triangle PML$, we have $(90° + \alpha) + x + y = 180°$. Finally,

$$\angle PLQ = \angle PLM + \angle MLN + \angle NLQ = y + \alpha + x = 180° - 90° = 90° \quad \blacksquare$$

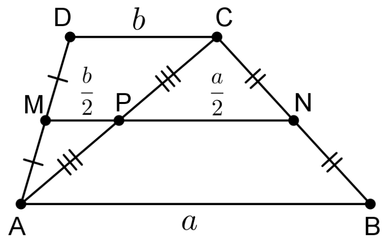
Example 2.6. Let $ABCD$ be a convex quadrilateral such that the side bisector of BC passes through the midpoint of AD and $\overline{AC} = \overline{BD}$. Prove that $\overline{AB} = \overline{CD}$.

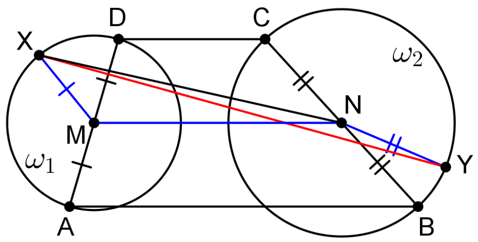


Proof. Let M be the midpoint of AD. Since M lies on the side bisector of BC, by Property 2.11, we get $\overline{MB} = \overline{MC}$. Now, since $\overline{MA} = \overline{MD}$, $\overline{AC} = \overline{DB}$ and $\overline{CM} = \overline{BM}$, by SSS, we get $\triangle MAC \cong \triangle MDB$. Therefore, $\angle MAC = \angle MDB$, i.e. $\angle DAC = \angle ADB$. Combining with $\overline{AC} = \overline{DB}$, $\overline{DA} = \overline{AD}$, by SAS, we get $\triangle DAC \cong \triangle ADB$. Therefore, $\overline{DC} = \overline{AB}$. ■

Example 2.7. In a trapezoid $ABCD$ with $AB \parallel CD$, ω_1 and ω_2 are two circles with diameters AD and BC, respectively. Let X and Y be two arbitrary points on ω_1 and ω_2, respectively. Show that the length of segment XY is not more than half the perimeter of $ABCD$.

Proof. Let M and N be the midpoints of AD and BC, respectively. Firstly, we will prove a property of the *midsegment in a trapezoid*, MN. Let P be the midpoint of AC. Then, MP is a midsegment in $\triangle ADC$, so $\overline{MP} = \frac{1}{2}\overline{DC}$ and $MP \parallel DC$. Similarly, PN is a midsegment in $\triangle ABC$, so $\overline{PN} = \frac{1}{2}\overline{AB}$ and $PN \parallel AB$. Therefore, $MP \parallel DC \parallel AB \parallel PN$, but since $P \in MP$ and $P \in PN$, then $MP \equiv PN$, i.e. it must be the same line, so $P \in MN$. So $\overline{MN} = \overline{MP} + \overline{PN} = \frac{1}{2}(\overline{DC} + \overline{AB})$.





Next, notice that since M is the midpoint of AD, the diameter of ω_1, then M is its center. Therefore, $\overline{MX} = \overline{MA} = \overline{MD} = \frac{1}{2}\overline{AD}$. Similarly, $\overline{NY} = \frac{1}{2}\overline{BC}$.

Finally, notice that $\overline{XM} + \overline{MN} + \overline{NY}$ is the length of a path from X to Y, which can not be shorter than the direct path from X to Y with length $\overline{XY}$. We can prove this more rigorously by using the Triangle Inequality in $\triangle XMN$ and $\triangle XNY$, i.e. $(\overline{XM} + \overline{MN}) + \overline{NY} \geq \overline{XN} + \overline{NY} \geq \overline{XY}$, from where we can get that $\overline{XY} \leq \overline{XM} + \overline{MN} + \overline{NY} = \frac{1}{2}(\overline{AD} + \overline{AB} + \overline{CD} + \overline{BC})$. ■

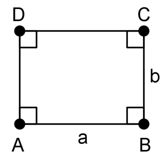
Now, try to solve the following related problems by yourself. They are located in the second part, Mixed Problems, at the end of the book.

Related problems: 1, 2, 3, 4, 5, 6, 9, 11, 12, 18, 20, 21, 23, 24 and 25.

Chapter 3

Area of Plane Figures

Rectangle

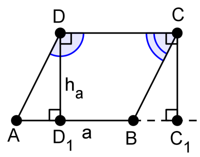


The area of a rectangle $ABCD$ is defined as the product of the length $a = \overline{AB} = \overline{CD}$ and the width $b = \overline{BC} = \overline{AD}$ of the rectangle.

$$P_{ABCD} = a \cdot b$$

Using this fact, we will derive the formulae for the area of other plane figures.

Parallelogram



Let $ABCD$ be a parallelogram. WLOG, let $\angle ABC > 90°$. Let C_1 and D_1 be the feet of the perpendiculars from C and D, respectively, to the line AB. Since $AD \parallel BC$, by Property 2.1, $\gamma = 180° - \delta$.

$$\angle BCC_1 = 90° - \gamma = \delta - 90° = \angle ADD_1$$

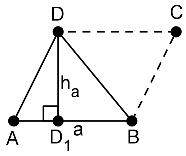
Additionally, $\overline{CC_1} = d(AB, CD) = \overline{DD_1}$ and $\angle CC_1B = 90° = \angle DD_1A$. Therefore, by the ASA criterion, $\triangle BCC_1 \cong \triangle ADD_1$. So $P_{\triangle BCC_1} = P_{\triangle ADD_1}$.

$$P_{ABCD} = P_{\triangle ADD_1} + P_{DD_1BC} = P_{\triangle BCC_1} + P_{DD_1BC} = P_{DD_1C_1C}$$

Since DD_1C_1C is a rectangle with length $\overline{CD} = \overline{AB} = a$ and width $\overline{CC_1} = h_a$, we get

$$P_{ABCD} = a \cdot h_a$$

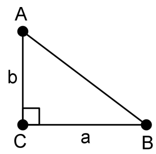
Triangle



Let $ABCD$ be a parallelogram. The diagonal BD divides the parallelogram in two triangles $\triangle ABD$ and $\triangle BCD$. By Property 2.15, the opposite sides of the parallelogram are equal, i.e. $\overline{AB} = \overline{CD}$ and $\overline{BC} = \overline{DA}$. Therefore, since $\angle BAD = 180° - \angle ADC = \angle DCB$, by the SAS criterion, $\triangle BAD \cong \triangle DCB$. Since congruent triangles have equal areas, then the area of each of the triangles is half the area of the parallelogram, i.e.

$$P_{\triangle ABD} = \frac{a \cdot h_a}{2}$$

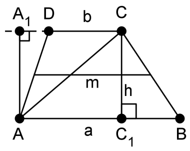
Right Triangle



In right triangle, the altitude opposite of the side a is in fact the side b, so

$$P_{\triangle ABC} = \frac{a \cdot b}{2}$$

Trapezoid



Let $ABCD$ be a trapezoid, such that $AB \parallel CD$. Let A_1 and C_1 be the feet of the altitudes from A and C to the lines CD and AB, respectively.

$$P_{ABCD} = P_{\triangle ABC} + P_{\triangle CDA} = \frac{\overline{AB} \cdot \overline{CC_1}}{2} + \frac{\overline{CD} \cdot \overline{AA_1}}{2}$$

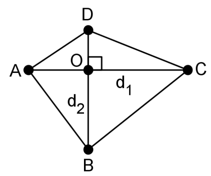
Let $h = d(AB, CD)$, $a = \overline{AB}$ and $b = \overline{CD}$. Then $\overline{AA_1} = \overline{CC_1} = h$. Therefore,

$$P_{ABCD} = \frac{a+b}{2} \cdot h$$

Since the midsegment in $ABCD$, m, is the sum of the midsegments in $\triangle ABC$ and $\triangle CDA$, the area of the trapezoid is sometimes expressed as

$$P_{ABCD} = m \cdot h$$

Quadrilateral with perpendicular diagonals



Let $ABCD$ be a quadrilateral with perpendicular diagonals. Let $AC \cap BD = O$. Then the triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$ and $\triangle DAO$ are right triangles. Therefore,

$$\begin{aligned} P_{ABCD} &= P_{\triangle ABO} + P_{\triangle BCO} + P_{\triangle CDO} + P_{\triangle DAO} = \\ &= \frac{\overline{AO} \cdot \overline{BO}}{2} + \frac{\overline{BO} \cdot \overline{CO}}{2} + \frac{\overline{CO} \cdot \overline{DO}}{2} + \frac{\overline{DO} \cdot \overline{AO}}{2} = \\ &= \frac{(\overline{AO} + \overline{CO}) \cdot (\overline{BO} + \overline{DO})}{2} = \frac{\overline{AC} \cdot \overline{BD}}{2} \end{aligned}$$

Let the diagonals AC and BD be d_1 and d_2, respectively. Then,

$$P_{ABCD} = \frac{d_1 \cdot d_2}{2}$$

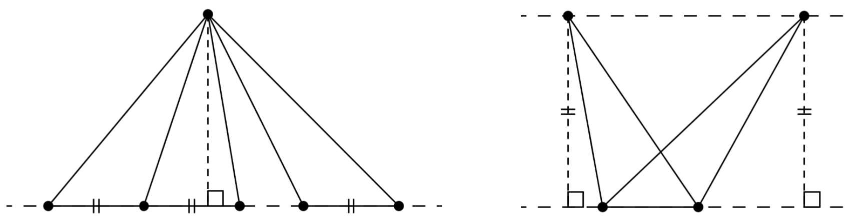
Area of Triangles

We will now show some properties that are often used in geometry problems.

Property 3.1.

(a) Two triangles that have base sides of equal length and a common altitude, have equal areas.

(b) Two triangles that have a common base side and altitudes of equal length, have equal areas.



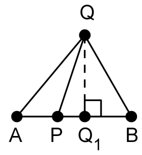
(a) Common altitude (b) Common base

Figure 3.1: Triangles with equal area

Proof. Follows directly by the formula for area of triangle $P_{\triangle ABC} = \dfrac{a \cdot h_a}{2}$. ■

Property 3.2. Let $A - P - B$ be collinear points in that order and let Q be a point that is not collinear with them. Then

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\overline{AP}}{\overline{PB}}.$$



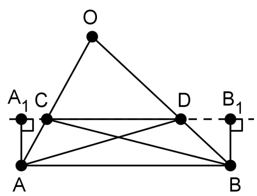
Proof. Let Q_1 be the foot of the perpendicular from Q to AB. Then,

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\frac{\overline{AP}\cdot\overline{QQ_1}}{2}}{\frac{\overline{PB}\cdot\overline{QQ_1}}{2}} = \frac{\overline{AP}}{\overline{PB}}. \qquad \blacksquare$$

We will use the proof of the following well-known theorem to present how these properties can be used.

Property 3.3 (Thales' Proportionality Theorem)**.** Let OAB be a triangle and let CD be a line that intersects its sides OA and OB at C and D, respectively. Prove that

$$AB \parallel CD \iff \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}}$$

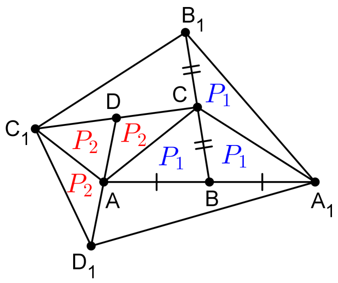


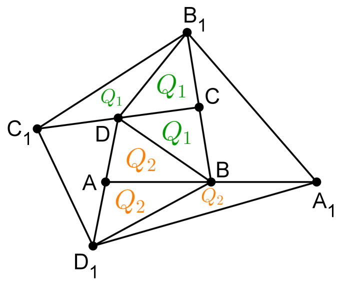
Proof. Let A_1 and B_1 be the feet of the perpendiculars from A and B, respectively, to the line CD. Then,

$$\begin{aligned} & AB \parallel CD \\ \iff\ & \overline{AA_1} = \overline{BB_1} \\ \overset{\text{Property 3.1}}{\iff}\ & P_{\triangle CDA} = P_{\triangle CDB} \\ \iff\ & \frac{P_{\triangle OCD}}{P_{\triangle CDA}} = \frac{P_{\triangle OCD}}{P_{\triangle CDB}} \\ \overset{\text{Property 3.2}}{\iff}\ & \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}} \end{aligned} \qquad \blacksquare$$

Now, we will solve a few problems using these properties.

Example 3.1. Let $ABCD$ be a convex quadrilateral with area 1. Let A_1 be a point on the ray AB beyond B, such that $\overline{AB} = \overline{BA_1}$. Similarly define the points B_1, C_1 and D_1. Prove that the area of $A_1B_1C_1D_1$ is 5.





Proof. Let P_1, P_2 denote the areas of $\triangle ABC, \triangle ADC$, respectively. The triangles $\triangle ABC$ and $\triangle BA_1C$ have base sides of equal length ($\overline{AB} = \overline{BA_1}$) and have a common altitude from C to the line AA_1. Therefore, by Property 3.1, we have $P_{\triangle BA_1C} = P_{\triangle ABC} = P_1$.

Similarly, the triangles $\triangle BCA_1$ and $\triangle CB_1A_1$ have base sides of equal length ($\overline{BC} = \overline{CB_1}$) and have a common altitude from A_1 to the line BB_1, so $P_{\triangle CB_1A_1} = P_{\triangle BCA_1} = P_1$.

Analogously, $P_{\triangle D_1AC_1} = P_{\triangle ADC_1} = P_{\triangle ADC} = P_2$. Therefore,

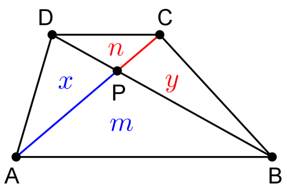
$$P_{\triangle BB_1A_1} + P_{\triangle DD_1C_1} = 2(P_1 + P_2) = 2P_{ABCD} = 2.$$

Similarly, if Q_1, Q_2 denote the area of $\triangle BCD, \triangle BAD$, respectively, we get

$$P_{\triangle CC_1B_1} + P_{\triangle AA_1D_1} = 2(Q_1 + Q_2) = 2P_{ABCD} = 2.$$

$$P_{A_1B_1C_1D_1} = P_{ABCD} + (P_{\triangle BB_1A_1} + P_{\triangle DD_1C_1}) + (P_{\triangle CC_1B_1} + P_{\triangle AA_1D_1}) = 5 \quad \blacksquare$$

Example 3.2. Let $ABCD$ be a trapezoid ($AB \parallel CD$). Let its diagonals AC and BD intersect at P. Let the areas of the triangle $\triangle ABP$ and $\triangle CDP$ be m and n, respectively. Prove that $P_{ABCD} = (\sqrt{m} + \sqrt{n})^2$.



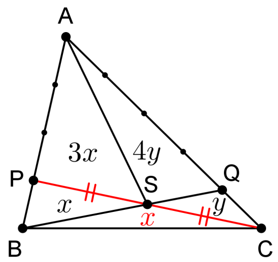
Proof. Let x, y denote the areas of $\triangle ADP, \triangle BCP$, respectively. The triangles $\triangle ABD$ and $\triangle ABC$ have a common base side AB and altitudes of equal length from D, C to AB since $d(D, AB) = d(CD, AB) = d(C, AB)$. Therefore, by Property 3.1, $P_{\triangle ABD} = P_{\triangle ABC}$, i.e. $x + m = y + m$, so $x = y$.

On the other hand, for the collinear points $A - P - C$ and for the points D, B that do not lie on AC, by Property 3.2, we have

$$\frac{P_{\triangle APD}}{P_{\triangle PCD}} = \frac{\overline{AP}}{\overline{PC}} = \frac{P_{\triangle APB}}{P_{\triangle PCB}}, \text{ i.e. } \frac{x}{n} = \frac{m}{y}, \text{ i.e. } mn = xy, \text{ i.e. } mn = x^2$$

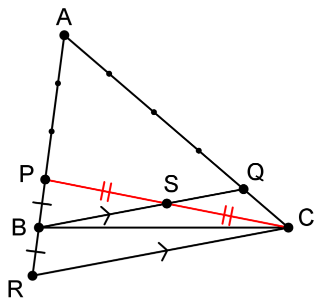
Finally, $P_{ABCD} = m + 2x + n = \sqrt{m}^2 + 2\sqrt{mn} + \sqrt{n}^2 = (\sqrt{m} + \sqrt{n})^2$. $\blacksquare$

Example 3.3. Let P be a point on the side AB in $\triangle ABC$, such that $\overline{AP} = 3 \cdot \overline{PB}$. Let $Q \in AC$, such that $\overline{AQ} = 4 \cdot \overline{QC}$. Prove that BQ bisects the line segment CP.



Proof 1. Let $BQ \cap CP = S$. We need to prove that $\overline{CS} = \overline{SP}$. By Property 3.1, that is equivalent to proving that $P_{\triangle CSB} = P_{\triangle SPB}$.

Let $x = P_{\triangle SPB}$. Since $\overline{AP} = 3 \cdot \overline{PB}$, by Property 3.2, we get $P_{\triangle SAP} = 3x$. Similarly, if $y = P_{\triangle SQC}$, since $\overline{AQ} = 4 \cdot \overline{QC}$, we get $P_{\triangle SAQ} = 4y$. But also, since $\overline{AQ} = 4 \cdot \overline{QC}$, we get $P_{\triangle BAQ} = 4P_{\triangle BQC}$, i.e. $x + 3x + 4y = 4(P_{\triangle CSB} + y)$. Therefore, $P_{\triangle CSB} = x = P_{\triangle SPB}$. ■



Proof 2. Again, let $BQ \cap CP = S$ and let R be a point on the extension of AB, such that $\overline{PB} = \overline{BR}$. Now, $\frac{\overline{AB}}{\overline{BR}} = \frac{4}{1} = \frac{\overline{AQ}}{\overline{QC}}$, so, by Thales' Proportionality Theorem, we get $BQ \parallel RC$. Focusing on $\triangle PRC$, we see that B is midpoint of PR and $BS \parallel RC$, so BS is midsegment, and thus S is the midpoint of PC. ■

Related problems: 8, 10 and 13.

Chapter 4

Similarity of Triangles

Two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are said to be similar when their corresponding angles are equal and their corresponding sides are proportional.

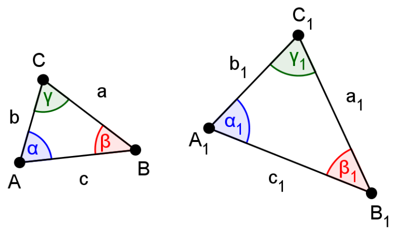


Figure 4.1: Similar triangles.

$$\triangle ABC \sim \triangle A_1B_1C_1 \iff \alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1, \frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c} = k$$

The positive real number k is called the *ratio of similarity.* If it is greater than 1, then $\triangle A_1B_1C_1$ is proportionally greater than $\triangle ABC$. If it is less than 1, then $\triangle A_1B_1C_1$ is proportionally smaller than $\triangle ABC$. If it is equal to 1, then $\triangle ABC$ and $\triangle A_1B_1C_1$ are congruent.

This ratio doesn't apply only for the lengths of the sides, but also for the lengths of other corresponding elements (for example, the length of an altitude, a median, etc). So, for the ratio of the areas of two similar triangles, we get:

$$\frac{P_1}{P} = \frac{\frac{a_1 \cdot h_{a_1}}{2}}{\frac{a \cdot h_a}{2}} = \frac{a_1}{a} \cdot \frac{h_{a_1}}{h_a} = k \cdot k = k^2 \quad \text{or} \quad k = \sqrt{\frac{P_1}{P}}.$$

There are also criteria for similarity of triangles.

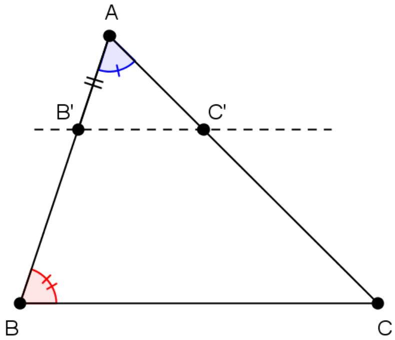
Criterion AA (angle-angle) If two pairs of corresponding angles are equal, then the triangles are similar.

Criterion SSS (side-side-side) If three pairs of corresponding sides are proportional, then the triangles are similar.

Criterion SAS (side-angle-side) If two pairs of corresponding sides are proportional and the angles between them are equal, then the triangles are similar.

We will now present the proofs of these criteria, for the sake of completeness. Although they use only the things that we learned until now, if you are a beginner, you may want to skip them (page 27) since the main point is to know how to use them. But if you are skeptical and don't believe that the criteria for similarity are really true, here are the proofs :)

Proof (AA). Let $\triangle ABC$ and $\triangle A_1B_1C_1$ be two trianges with $\alpha = \alpha_1$ and $\beta = \beta_1$. By Property 2.2, $\gamma = \gamma_1$, too. WLOG, let $\overline{A_1B_1} < \overline{AB}$. Then, we



can construct a point $B' \in AB$, such that $\overline{AB'} = \overline{A_1B_1}$. The parallel line to BC through B' intersects AC at C'. Then, by Property 2.1, $\angle AB'C' = \angle ABC$. So, by the ASA criterion for congruent triangles, we have $\triangle AB'C' \cong \triangle A_1B_1C_1$.

Since $BC \parallel B'C'$, by Thales' Proportionality Theorem, we have

$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}}.$$

$$\frac{\overline{B'B}}{\overline{AB'}} = \frac{\overline{C'C}}{\overline{AC'}}$$

$$\frac{\overline{B'B}}{\overline{AB'}} + 1 = \frac{\overline{C'C}}{\overline{AC'}} + 1$$

$$\frac{\overline{B'B} + \overline{AB'}}{\overline{AB'}} = \frac{\overline{C'C} + \overline{AC'}}{\overline{AC'}}$$

$$\frac{\overline{AB}}{\overline{AB'}} = \frac{\overline{AC}}{\overline{AC'}}$$

$$\frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}}$$

Now, by substituting the correpsonding sides from the congruence we just proved, we get

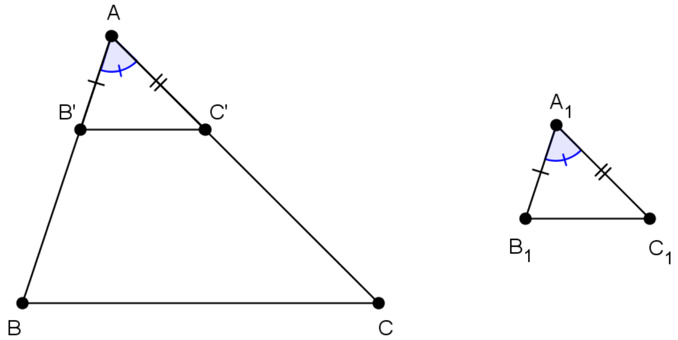
$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}}.$$

Similarly, by constructing a point $A'' \in BA$ and then a line $A''C''$ that is parallel to AC, we can get that

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{B_1C_1}}{\overline{BC}}.$$

Therefore, all the three corresponding angles are equal and the three corresponding pairs of sides are proportional, so $\triangle ABC \sim \triangle A_1B_1C_1$. ∎

Proof (SAS). Let $\triangle ABC$ and $\triangle A_1B_1C_1$ be two triangles with $\alpha = \alpha_1$ and $\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = k$.



WLOG, let $k < 1$. Then, we can construct points $B' \in AB$ and $C' \in AC$, such that $\overline{AB'} = \overline{A_1B_1}$ and $\overline{AC'} = \overline{A_1C_1}$. By substituting the line segments with equal lengths, we get

$$k = \frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}}.$$

Similarly as in the previous proof, by algebraic transformations (taking the reciprocal value, subtracting 1 on both sides, and taking the reciprocal value once again), we get

$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}},$$

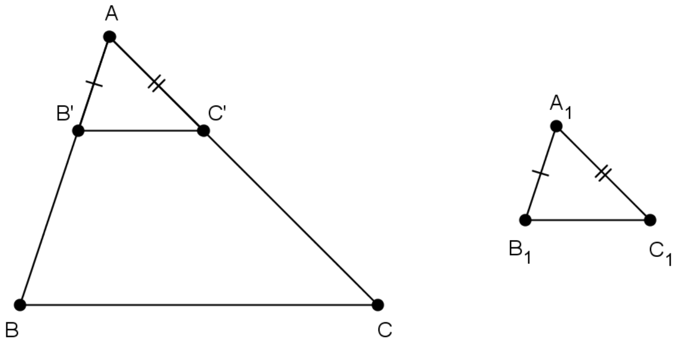
which by Thales' Proportionality Theorem means that $B'C' \parallel BC$. Therefore, by Property 2.1, we get that $\angle AB'C' = \angle ABC$ and $\angle AC'B' = \angle ACB$.

By the SAS criterion for congruence, we get $\triangle AB'C' \cong \triangle A_1B_1C_1$. Therefore, $\angle AB'C' = \angle A_1B_1C_1$ and $\angle AC'B' = \angle A_1C_1B_1$. By combining this with the previous result, we get that $\beta = \beta_1$ and $\gamma = \gamma_1$.

In conclusion, all the angles in the triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are equal, so by the criterion AA that we previously proved, we get that $\triangle ABC \sim \triangle A_1B_1C_1$. ∎

Proof (SSS). Let $\triangle ABC$ and $\triangle A_1B_1C_1$ be two triangles with

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = \frac{\overline{B_1C_1}}{\overline{BC}} = k.$$



WLOG, let $k < 1$. Then, we can construct points $B' \in AB$ and $C' \in AC$, such that $\overline{AB'} = \overline{A_1B_1}$ and $\overline{AC'} = \overline{A_1C_1}$. Therefore, we have

$$k = \frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}},$$

which, as in the previous proof, by algebraic transformations (taking the reciprocal value, subtracting 1 on both sides, and taking the reciprocal value once again), becomes

$$\frac{\overline{AB'}}{\overline{B'B}} = \frac{\overline{AC'}}{\overline{C'C}}.$$

Therefore, by Thales' Proportionality Theorem, $B'C' \parallel BC$, so by Property 2.1, $\angle AB'C' = \angle ABC$ and $\angle AC'B' = \angle ACB$. By the AA criterion that we earlier proved, we get that $\triangle AB'C' \sim \triangle ABC$ and therefore

$$\frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{BC}}.$$

By substituting the line segments with equal length that we constructed, we get

$$\frac{\overline{A_1B_1}}{\overline{AB}} = \frac{\overline{A_1C_1}}{\overline{AC}} = \frac{\overline{B'C'}}{\overline{BC}}.$$

Combining this with the condition, we can conclude that

$$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{B_1C_1}}{\overline{BC}}, \text{ i.e. } \overline{B'C'} = \overline{B_1C_1}.$$

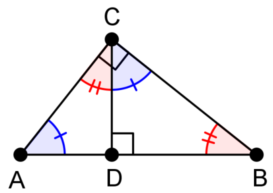
Now, by the SSS criterion for congruence, we get that $\triangle AB'C' \cong \triangle A_1B_1C_1$ and therefore $\angle B'AC' = \angle B_1A_1C_1$. But $\angle B'AC' \equiv \angle BAC$, so $\angle B_1A_1C_1 = \angle BAC$. Combining this with the condition, by the SAS criterion for similarity that we earlier proved, we get that $\triangle A_1B_1C_1 \sim \triangle ABC$. ∎

Property 4.1 (Euclid's laws)**.** In a right triangle ABC, with the right angle at C, let D be the foot of the perpendicular from C to AB. Prove that:

$$\overline{CD}^2 = \overline{AD} \cdot \overline{DB}$$

$$\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$$

$$\overline{BC}^2 = \overline{BD} \cdot \overline{BA}.$$



Proof. Let $\angle CAB = \alpha$ and $\angle CBA = \beta$. Since $\angle ACB = 90°$ and we know that all the angles in a triangle add up to $180°$, then $\alpha + \beta = 90°$. Now looking at the triangles ACD and BCD, and remembering again the sum of angles in a triangle, we get that $\angle ACD = 180° - 90° - \alpha = \beta$ and $\angle BCD = 180° - 90° - \beta = \alpha$.

$$\triangle ADC \sim \triangle CDB \text{ (by the criterion AA)}$$

$$\therefore \frac{\overline{AD}}{\overline{DC}} = \frac{\overline{CD}}{\overline{DB}}, \text{ i.e. } \overline{CD}^2 = \overline{AD} \cdot \overline{DB}$$

$$\triangle ACD \sim \triangle ABC \text{ (by the criterion AA)}$$

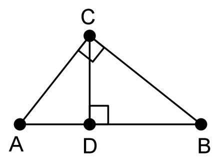
$$\therefore \frac{\overline{AC}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AC}}, \text{ i.e. } \overline{AC}^2 = \overline{AD} \cdot \overline{AB}$$

$$\triangle BCD \sim \triangle BAC \text{ (by the criterion AA)}$$

$$\therefore \frac{\overline{BC}}{\overline{BD}} = \frac{\overline{BA}}{\overline{BC}}, \text{ i.e. } \overline{BC}^2 = \overline{BD} \cdot \overline{BA}$$

■

Property 4.2 (Pythagorean Theorem)**.** Prove that the square of the hypothenuse in a right triangle is equal to the sum of the squares of the legs.



Proof. Let ABC be a right triangle with right angle at C and let CD be an altitude in that triangle. From Property 4.1, we know that $\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$ and $\overline{BC}^2 = \overline{BD} \cdot \overline{BA}$. By adding these equations, we get

$$\overline{AC}^2 + \overline{BC}^2 = \overline{AB} \cdot (\overline{AD} + \overline{BD}) = \overline{AB}^2$$

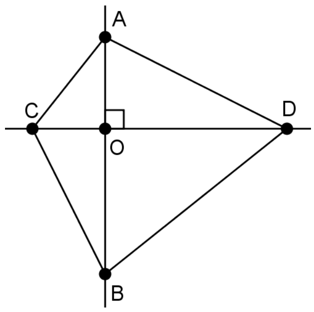
■

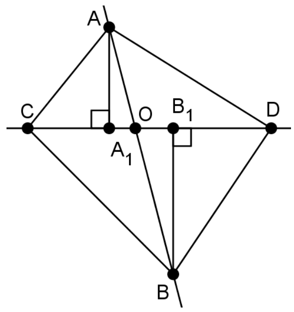
Property 4.3. Let AB and CD be two intersecting lines. Then,

$$AB \perp CD \iff \overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2.$$

Proof. Let $AB \cap CD = O$. We will firstly prove the first direction, so let $AB \perp CD$. Then, the triangles $\triangle ACO$, $\triangle BCO$, $\triangle ADO$ and $\triangle BDO$ are right triangles, so by the Pythagorean Theorem, we get

$$\begin{aligned}\overline{CA}^2 - \overline{CB}^2 &= (\overline{OC}^2 + \overline{OA}^2) - (\overline{OC}^2 + \overline{OB}^2) = \overline{OA}^2 - \overline{OB}^2 = \\ &= (\overline{OD}^2 + \overline{OA}^2) - (\overline{OD}^2 + \overline{OB}^2) = \overline{DA}^2 - \overline{DB}^2 \quad \square\end{aligned}$$





Now, let's prove the other direction. Let $\overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2$. We will discuss the case where O is between A and B and between C and D. Let the feet of the perpendiculars from A and B to CD be A_1 and B_1, respectively. Then, triangles $\triangle CAA_1$, $\triangle CBB_1$, $\triangle DAA_1$ and $\triangle DBB_1$ are right triangles, so by using the Pythagorean Theorem and substituting in the condition, we get

$$(\overline{CA_1}^2 + \overline{AA_1}^2) - (\overline{CB_1}^2 + \overline{BB_1}^2) = (\overline{DA_1}^2 + \overline{AA_1}^2) - (\overline{DB_1}^2 + \overline{BB_1}^2)$$

After canceling on both sides, we get

$$\overline{CA_1}^2 - \overline{CB_1}^2 = \overline{DA_1}^2 - \overline{DB_1}^2$$
$$\overline{CA_1}^2 - \overline{DA_1}^2 = \overline{CB_1}^2 - \overline{DB_1}^2$$

Using the formula for difference of squares, we get

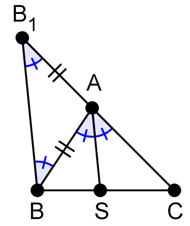
$$(\overline{CA_1} - \overline{DA_1}) \cdot (\overline{CA_1} + \overline{DA_1}) = (\overline{CB_1} - \overline{DB_1}) \cdot (\overline{CB_1} + \overline{DB_1})$$
$$(\overline{CA_1} - \overline{DA_1}) \cdot \overline{CD} = (\overline{CB_1} - \overline{DB_1}) \cdot \overline{CD}$$
$$\overline{CA_1} - \overline{CB_1} = \overline{DA_1} - \overline{DB_1}$$
$$-\overline{A_1B_1} = \overline{A_1B_1}$$
$$0 = 2 \cdot \overline{A_1B_1}$$
$$A_1 \equiv B_1$$

Therefore, the perpendiculars to CD from A and B pass through a common point on CD, so they must be the same line, i.e. $AB \perp CD$.

In the cases where O is not between A and B or between C and D, the proof follows exactly the same steps. There might be a different operation when dealing with the line segments (addition or subtraction) depending on the configuration, but the result will always be the same. ∎

Property 4.4 (Angle Bisector Theorem)**.** The angle bisector in a triangle divides the opposite side in segments proportional to the other two sides of the triangle.

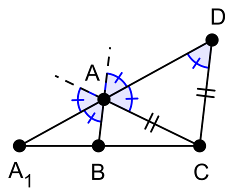
Proof. Here is an idea how to prove this theorem. Let ABC be a triangle and let S be a point on BC, such that AS is an angle bisector in $\triangle ABC$. We need to prove that $\frac{\overline{BS}}{\overline{SC}} = \frac{\overline{AB}}{\overline{AC}}$. If we rearrange this equality, we get that we need to prove that $\frac{\overline{CS}}{\overline{SB}} = \frac{\overline{CA}}{\overline{AB}}$. This resembles the Thales' Proportionality Theorem, with the exception that the points C, A and B are not collinear. So if we take a point B_1 on the extension of CA, such that $\overline{AB_1} = \overline{AB}$, then we will only need to prove that SA is parallel to BB_1.



Let $B_1 \in CA$, such that $\overline{AB_1} = \overline{AB}$. The triangle $\triangle ABB_1$ is isosceles, so $\angle ABB_1 = \angle AB_1B = \varphi$. The angle $\angle BAC$ is exterior angle of $\triangle ABB_1$, so $\angle BAC = \angle ABB_1 + \angle AB_1B = 2\varphi$. Since AS is an angle bisector, $\angle BAS = \frac{1}{2}\angle BAC = \varphi$. So, $\angle BAS = \angle ABB_1$, which means that $SA \parallel BB_1$. By the Thales' Proportionality Theorem, we get that $\frac{\overline{CS}}{\overline{SB}} = \frac{\overline{CA}}{\overline{AB_1}}$. By substituting $\overline{AB}$ for $\overline{AB_1}$, we get $\frac{\overline{CS}}{\overline{SB}} = \frac{\overline{CA}}{\overline{AB}}$. ■

Property 4.5 (External Angle Bisector Theorem)**.** Let the bisector of the exterior angle at vertex A in $\triangle ABC$ intersect the line BC at A_1. Prove that

$$\frac{\overline{BA_1}}{\overline{A_1C}} = \frac{\overline{AB}}{\overline{AC}}.$$



Proof. WLOG, let $\overline{AB} < \overline{AC}$, i.e. $\overline{A_1B} < \overline{A_1C}$. Let D be a point on the line AA_1, such that $AB \parallel CD$. Then, by Property 2.1, $\angle A_1AB = \angle A_1DC$, so $\triangle A_1AB \sim \triangle A_1DC$ and therefore

$$\frac{\overline{A_1B}}{\overline{A_1C}} = \frac{\overline{AB}}{\overline{DC}}. \tag{$\star$}$$

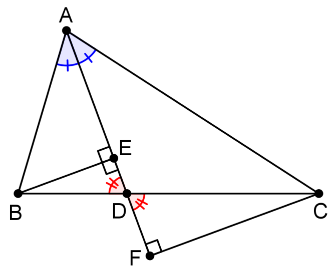
Let α' be the external angle at the vertex A in $\triangle ABC$. Then, as vertical angles,

$$\angle DAC = \frac{\alpha'}{2} = \angle A_1AB = \angle A_1DC \equiv \angle ADC,$$

so $\triangle ADC$ is isosceles, i.e. $\overline{AC} = \overline{DC}$. By substituting in ($\star$), we get the desired ratio. ■

Example 4.1. In the triangle ABC, let BE and CF be perpendiculars to the angle bisector AD. Prove that $\overline{AE} \cdot \overline{DF} = \overline{AF} \cdot \overline{DE}$.

Proof. Whenever we have to prove products of lengths that are equal, it is good to try and rearrange the terms in order to get ratios of lengths that can be considered as part of a similarity of triangles. The product above can be rearranged to $\frac{\overline{AE}}{\overline{AF}} = \frac{\overline{DE}}{\overline{DF}}$. As all of this points are collinear, these four lengths can not all be sides of a pair of similar triangles, so we can not directly solve this with 1 pair of similar triangles. However, each of the sides can represent the ratio od similarity of each of 2 pairs of similar triangles.



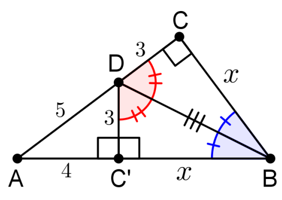
The first pair of similar triangles is $\triangle AEB \sim \triangle AFC$. They are similar by the criterion AA, since $\angle BAE = \frac{\alpha}{2} = \angle CAF$ and $\angle AEB = 90^\circ = \angle AFC$. Thus, $\frac{\overline{AE}}{\overline{AF}} = \frac{\overline{BE}}{\overline{CF}}$.

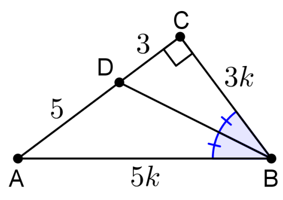
The second pair of similar triangles is $\triangle BED \sim \triangle CFD$. They are also similar by the criterion AA, since $\angle BED = 90^\circ = \angle CFD$ and $\angle BDE = \angle CDF$ as vertical angles. Thus, $\frac{\overline{BE}}{\overline{CF}} = \frac{\overline{DE}}{\overline{DF}}$.

$$\therefore \frac{\overline{AE}}{\overline{AF}} = \frac{\overline{BE}}{\overline{CF}} = \frac{\overline{DE}}{\overline{DF}}, \text{ i.e. } \overline{AE} \cdot \overline{DF} = \overline{AF} \cdot \overline{DE} \qquad \blacksquare$$

Example 4.2. Let $\triangle ABC$ be a right triangle ($\gamma = 90^\circ$). The angle bisector of $\angle ABC$ intersects AC at D. If $\overline{AD} = 5$ and $\overline{CD} = 3$, find $\overline{AB}$.

Proof 1. Let C' be the foot of the perpendicular from D to AB. Now, we focus on $\triangle BDC$ and $\triangle BDC'$. We have $\angle CBD = \frac{\beta}{2} = \angle C'BD$, BD is a common side and $\angle BDC = 90^\circ - \frac{\beta}{2} = \angle BDC'$, so by the criterion ASA for congruent triangles, we get $\triangle BDC \cong \triangle BDC'$. Therefore, $\overline{DC'} = \overline{DC} = 3$ and $\overline{BC'} = \overline{BC} = x$. Now, from the right $\triangle ADC'$, by Pythagorean Theorem, we get $\overline{AD}^2 = \overline{DC'}^2 + \overline{C'A}^2$, i.e. $\overline{C'A} = 4$. Finally, from the right $\triangle ABC$, we get $\overline{AB}^2 = \overline{BC}^2 + \overline{CA}^2$, i.e. $(4+x)^2 = x^2 + 8^2$ which has a solution $x = 6$. Therefore, $\overline{AB} = \overline{AC'} + \overline{C'B} = 4 + 6 = 10$. $\blacksquare$





Proof 2. From Angle Bisector Theorem in $\triangle ABC$, we get $\frac{\overline{BA}}{\overline{BC}} = \frac{\overline{DA}}{\overline{DC}} = \frac{5}{3}$. Therefore, $\overline{BA} = 5k$ and $\overline{BC} = 3k$. Now, using Pythagorean Theorem in $\triangle ABC$, we get $(5k)^2 = (3k)^2 + 8^2$, which has a positive solution $k = 2$, i.e. $\overline{AB} = 5k = 10$. $\blacksquare$

Related problems: 22 and 28.

Chapter 5

Circles

A circle is a set of point equidistant from one previously chosen point, called the center. The distance from the center to the circle is called the radius of the circle. We will usually notate a circle with center O and radius r as $\omega(O, r)$.

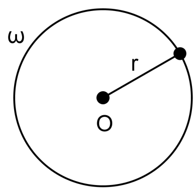
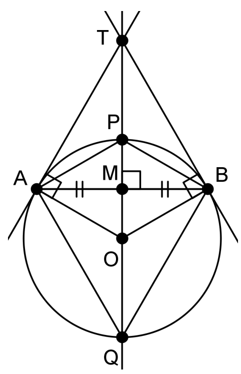


Figure 5.1: Circle ω with center O and radius r.

Symmetry in a Circle



Let AB be a chord in a circle. If we connect the points A and B with the center O, we get an isosceles triangle ABO. If M is the midpoint of AB, then by SSS $\triangle AMO \cong \triangle BMO$ and therefore $\angle AMO = \angle BMO$, i.e. $OM \perp AB$. Also, $\angle AOM = \angle BOM$, so if we denote by P and Q the intersections of OM with the circle, we get that $\triangle OAP \cong \triangle OBP$ (by SAS) which yields $\overline{AP} = \overline{BP}$ and consequently $\overparen{AP} = \overparen{BP}$. Similarly, $\triangle AOQ \cong \triangle BOQ$ (by SAS) and $\overparen{AQ} = \overparen{BQ}$. Looking from a different perspective, this all means that the center of the circle O and the midpoints of the minor and major arc $\overparen{AB}$, P and Q, all lie on the perpendicular bisector of the chord AB. Hence, the center of any circle can be found as the intersection of the perpendicular bisectors of any two chords.

Moreover, let T be the intersection of the tangents at A and B. By the Pythagorean Theorem, $\overline{TA}^2 = \overline{TO}^2 - \overline{OA}^2 = \overline{TO}^2 - \overline{OB}^2 = \overline{TB}^2$, i.e. $\overline{TA} = \overline{TB}$. So the tangent segments from a point to the circle are equal. Now, by SSS $\triangle OAT \cong \triangle OBT$, so $\angle TOA = \angle TOB$, which combined with the previous findings, means that T also lies on the perpendicular bisector of the chord AB.

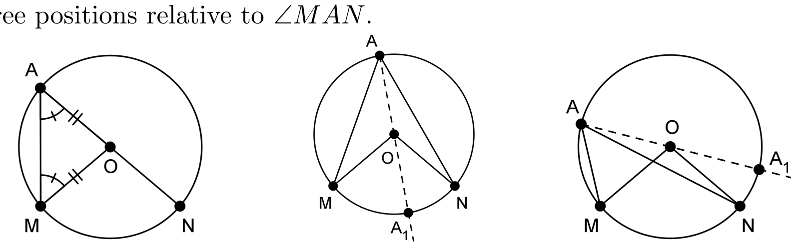
Angles in a Circle

An inscribed angle is an angle whose vertex lies on a circle and its rays intersect that circle.

A central angle is an angle whose vertex is the center of the circle and its rays intersect that circle.

(a) Inscribed angle (b) Central angle

Now, let's take a look at the relation between an inscribed angle and a central angle that subtend the same arc. Let $\angle MAN$ and $\angle MON$ be an inscribed and the central angle that subtend the arc $\widehat{MN}$, respectively. The center O can be in three positions relative to $\angle MAN$.



i) O lies on one of the rays of $\angle MAN$, WLOG let O lie on the ray AN.

$$\overline{OA} = r = \overline{OM}$$

$$\therefore \triangle OAM \text{ is isosceles.}$$

$$\therefore \angle OAM = \angle OMA$$

$$\therefore \angle MON = \angle OAM + \angle OMA = 2 \cdot \angle OAM \equiv 2 \cdot \angle MAN$$

ii) O is in the interior of $\angle MAN$.
Let A_1 be the second intersection of AO with the circle.

$$\angle MOA_1 = 2 \cdot \angle MAA_1 \text{ (from case i)}$$

$$\angle A_1ON = 2 \cdot \angle A_1AN \text{ (from case i)}$$

$$\therefore \angle MON = \angle MOA_1 + \angle A_1ON = 2 \cdot \angle MAA_1 + 2 \cdot \angle A_1AN = 2 \cdot \angle MAN$$

iii) O is in the exterior of $\angle MAN$, WLOG O is closer to the ray AN.
Let A_1 be the second intersection of AO with the circle.

$$\angle MOA_1 = 2 \cdot \angle MAA_1 \text{ (from case i)}$$

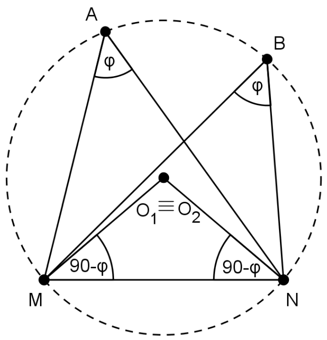
$$\angle NOA_1 = 2 \cdot \angle NAA_1 \text{ (from case i)}$$

$$\therefore \angle MON = \angle MOA_1 - \angle NOA_1 = 2 \cdot \angle MAA_1 - 2 \cdot \angle NAA_1 = 2 \cdot \angle MAN$$

Therefore, any inscribed angle is half the central angle that subtends the same arc. It also implies that all the inscribed angles that subtend the same arc are equal.

The converse is also true. The proof is "less attractive", but it will be presented for the sake of completeness :) We will prove that if two angles $\angle MAN$ and $\angle MBN$ are equal (and their vertices A and B lie on the same side of the line MN), then their vertices, A and B, and the intersection points of their corresponding rays, M and N, are concyclic.

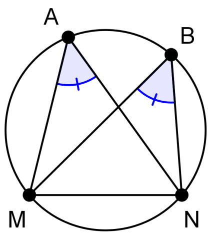
Let $\omega_1(O_1, r_1)$ be the circumcircle of $\triangle MAN$. Let $\varphi = \angle MAN = \angle MBN$. Therefore, $\angle MO_1N = 2 \cdot \angle MAN = 2\varphi$. Since $\triangle MO_1N$ is isosceles (because $\overline{O_1M} = r_1 = \overline{O_1N}$), $\angle O_1MN = \angle O_1NM = 90^\circ - \varphi$. Similiarly, if $\omega_2(O_2, r_2)$ is the circumcirle of $\triangle MBN$, then $\angle O_2MN = \angle O_2NM = 90^\circ - \varphi$. Therefore, by the ASA criterion, $\triangle MO_1N \cong \triangle MO_2N$. Since A and B, and consequently O_1 and O_2 lie on the same side of MN, we get that $O_1 \equiv O_2$. Therefore, $r_1 = \overline{O_1M} = \overline{O_2M} = r_2$, so $\omega_1 \equiv \omega_2$, i.e. the points M, A, B and N lie on a single circle.



In conclusion, we get two important properties of the angles in a circle:

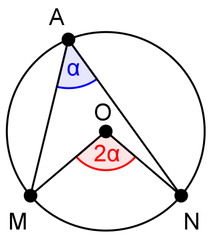
Property 5.1. Inscribed angles that subtend the same arc are equal. The converse is also true, i.e. if two angles are equal, then their vertices and the intersection points of their corresponding rays are concyclic.

$$M, A, B, N \in \omega \text{ (in that order)} \iff \angle MAN = \angle MBN \tag{5.1}$$



Property 5.2. The central angle is twice an inscribed angle that subtends the same arc.

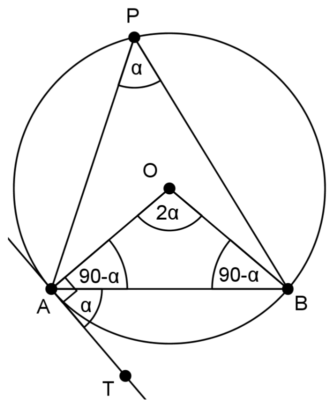
$$M, A, N \in \omega(O, r) \implies \angle MON = 2 \cdot \angle MAN \tag{5.2}$$



Remark. Be careful, if for a point P we have $\angle MPN = 2 \cdot \angle MAN$, it doesn't mean that $P \equiv O$ because all angles $\angle MXN$ with X on arc $\widehat{MON}$ are equal.

Finally, let's investigate the angle between a tangent and a chord through the tangent point.

Let AB be a chord in $\omega(O, r)$ and let TA be a tangent to ω at A. Let $\angle BAT = \alpha$. Since TA is a tangent, then it must be perpendicular to OA, i.e. $\angle OAT = 90°$.



$$\therefore \angle OAB = \angle OAT - \angle BAT = 90° - \alpha$$

$$\overline{OA} = r = \overline{OB}$$

$$\therefore \triangle OAB \text{ is isosceles}$$

$$\therefore \angle OAB = \angle OBA = 90° - \alpha$$

$$\angle AOB = 180° - 2(90° - \alpha) = 180° - 180° + 2\alpha = 2\alpha$$

Let $\angle APB$ be any inscribed angle over the arc $\overset{\frown}{AB}$. Then,

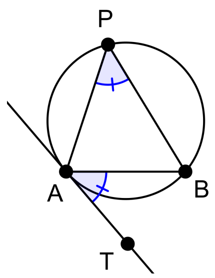
$$\angle APB \overset{(5.2)}{=} \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 2\alpha = \alpha.$$

In conclusion, we get the following property:

Property 5.3. The angle between a tangent and a chord is equal to any inscribed angle that subtends that chord.

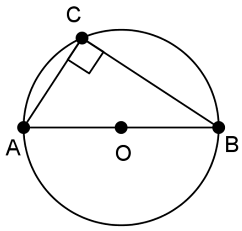
$$\angle TAB = \angle APB \tag{5.3}$$

The converse is also true, i.e. if an angle between a chord and a line through one of the endpoints of the chord is equal to an inscribed angle that subtends that chord, then that line must be tangent to the circle.



We will now see a few useful consequences of the relation between an inscribed and a central angle.

Property 5.4 (Thales' Theorem)**.** Every inscribed angle that subtends a diameter is a right angle.

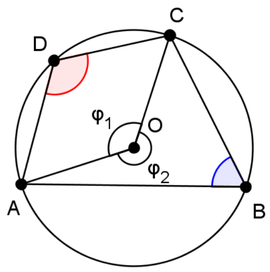


Proof. Let AB be a diameter in a circle with center O, and let C be another point on the circle.

$$\angle ACB \stackrel{(5.2)}{=} \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 180^\circ = 90^\circ \qquad \blacksquare$$

Remark. Moreover, we can see that inscribed angles that subtend an arc greater than half the circumference are obtuse and inscribed angles that subtend an arc smaller than half the circumference are acute.

Property 5.5. The opposite angles of a cyclic quadrilateral are supplementary.



Proof. Let $ABCD$ be a cyclic quadrilateral and let its circumcircle be centered at O. Let φ_1 and φ_2 be the central angles that subtend the arcs $\overset{\frown}{ADC}$ and $\overset{\frown}{ABC}$, respectively.

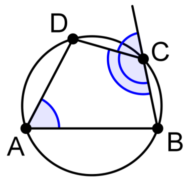
$$\angle ABC \stackrel{(5.2)}{=} \frac{1}{2}\varphi_1 \text{ (over the arc } \overset{\frown}{ADC}\text{)}$$

$$\angle ADC \stackrel{(5.2)}{=} \frac{1}{2}\varphi_2 \text{ (over the arc } \overset{\frown}{ABC}\text{)}$$

$$\therefore \angle ABC + \angle ADC = \frac{1}{2} \cdot (\varphi_1 + \varphi_2) = \frac{1}{2} \cdot 360^\circ = 180^\circ \qquad \blacksquare$$

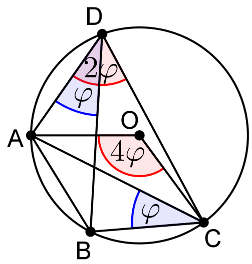
We will now solve a few examples to see how these 5 properties can be used.

Example 5.1. Let $ABCD$ be a cyclic quadrilateral. Prove that the external angle at C is equal to the internal angle at A.



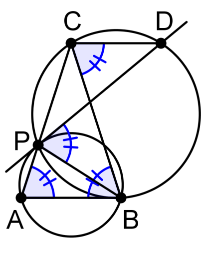
Proof. For the external angle γ_1, we have $\gamma + \gamma_1 = 180°$. From Property 5.5, we know that $\alpha + \gamma = 180°$. Therefore, $\gamma_1 = 180° - \gamma = \alpha$. ■

Example 5.2. Let O be the circumcenter of the cyclic quadrilateral $ABCD$, such that the points B, C lie on one side of the line AO, while D lies on the other side. If $\angle AOC = 4 \cdot \angle ACB$, prove that DB is the angle bisector of $\angle ADC$.



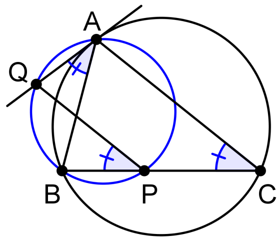
Proof. Let $\varphi = \angle ACB$. Then, $\angle AOC = 4\varphi$. From Property 5.2, we get that $\angle ADC = \frac{1}{2}\angle AOC = 2\varphi$. From Property 5.1, we get that $\angle ADB = \angle ACB = \varphi$. Finally, $\angle BDC = \angle ADC - \angle ADB = 2\varphi - \varphi = \varphi = \angle ADB$, so DB is angle bisector of $\angle ADC$. ■

Example 5.3. Let ABC be an isosceles triangle, such that $\overline{AC} = \overline{BC}$. Let P be a point on the side AC. The tangent to (ABP) at the point P intersects (BCP) at D. Prove that $CD \parallel AB$.



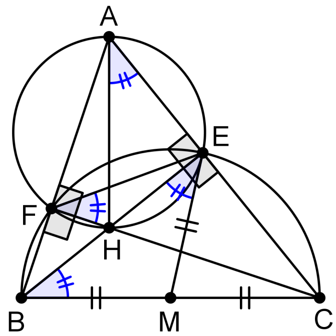
Proof. By Property 2.1, we need to prove that $\angle DCB = \angle CBA$. Since the points D, C, P, B lie on the same circle, by Property 5.1, we get $\angle DCB = \angle DPB$. Since DP is tangent to (ABP), by Property 5.3, we get $\angle DPB = \angle PAB$. Finally, since $\triangle CAB$ is isosceles, we get $\angle CAB = \angle CBA$. Therefore, $\angle DCB = \angle DPB = \angle PAB \equiv \angle CAB = \angle CBA$. ■

Example 5.4. Let P be a point on the side BC of $\triangle ABC$. The parallel line to AC through P intersects the tangent to (ABC) through A at a point Q. Prove that $APBQ$ is cyclic.



Proof. By the converse of Property 5.1, we need to prove that $\angle QAB = \angle QPB$. From Property 5.3, we get $\angle QAB = \angle ACB$. From Property 2.1, we get $\angle ACB = \angle QPB$. Therefore, $\angle QAB = \angle QPB$. ■

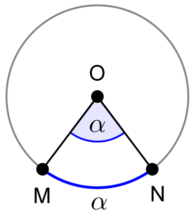
Example 5.5. Let ABC be an acute triangle. Let E and F be the feet of the altitudes in $\triangle ABC$ from B and C, respectively, and let M be the midpoint of BC. Prove that ME and MF are tangents to (AEF).



Proof. Let $BE \cap CF = H$. Since $\angle HEA + \angle HFA = 90° + 90° = 180°$, by the converse of Property 5.5, we get that $AEHF$ is cyclic quadrilateral. In order to prove that ME is tangent to $(AEHF)$, by the converse of Property 5.3, we need to prove that $\angle MEH = \angle EAH$.

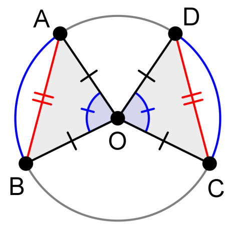
Since $AEHF$ is cyclic, by Property 5.1, we get $\angle EAH = \angle EFH$. Since $\angle BEC = 90° = \angle BFC$, by the converse of Thales' Theorem, E, F lie on the circle with diameter BC, so $BFEC$ is cyclic. Thus, by Property 5.1, we get $\angle EFC = \angle EBC$. Since M is midpoint of the diameter BC, we get that M is the center of $(BFEC)$, so $\overline{MB} = \overline{ME}$ as radii. Therefore, $\angle MBE = \angle MEB$. Thus, $\angle EAH = \angle EFH \equiv \angle EFC = \angle EBC \equiv \angle EBM = \angle MEB \equiv \angle MEH$, so ME is tangent to (AEF). Similarly, MF is tangent to (AEF). ■

For what follows, it would be better if we introduced the notation of *angle measure of an arc*. Namely, it corresponds to the central angle that subtends that arc, i.e. we will say that $\widehat{MN} = \alpha$ if and only if $\angle MON = \alpha$. Therefore, a semicircle has a measure of $180°$ and the whole circle has an angle measure of $360°$. Firstly, let's prove two simple properties about arcs in circles and later we will continue with some more angles in a circle.



Property 5.6. Let A, B, C, D be points on a circle ω. Then,

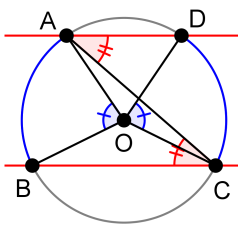
$$\widehat{AB} = \widehat{CD} \iff \overline{AB} = \overline{CD}.$$



Proof. Let O be the center of ω. Then, $\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD} = r$, so $\widehat{AB} = \widehat{CD} \iff \angle AOB = \angle COD \overset{SAS}{\iff} \triangle AOB \cong \triangle COD \overset{SSS}{\iff} \overline{AB} = \overline{CD}$ ■

Property 5.7. Let A, B, C, D be points on a circle ω. Then,

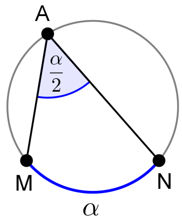
$$\widehat{AB} = \widehat{CD} \iff BC \parallel AD.$$



Proof. Let O be the center of ω. Then,
$\widehat{AB} = \widehat{CD} \iff \angle AOB = \angle COD \overset{(5.2)}{\iff} \angle ACB = \angle CAD \overset{(2.1)}{\iff} BC \parallel AD$ ■

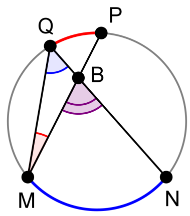
Remark. Can you see how one can use this property to quickly draw an accurate diagram in Problem 150 (points M and N), Problem 158 (points D and E) and Problem 201 (points X and Y)?

Notice that, by Equation 5.2, for an inscribed angle $\angle MAN$, we have $\angle MAN = \frac{\widehat{MN}}{2}$. Now, let's see what happens when the vertex of the angle is an arbitrary point inside or outside the circle.



Property 5.8 (Interior angle in a circle)**.** Let B be a point in the interior of a circle ω, such that the chords MP and NQ intersect at B. Then, for the *interior angle* $\angle MBN$, we have:

$$\angle MBN > \frac{\widehat{MN}}{2} \quad \text{and} \quad \angle MBN = \frac{\widehat{MN} + \widehat{PQ}}{2}.$$



Proof. As an exterior angle in $\triangle MBQ$, we have

$$\angle MBN = \angle MQB + \angle BMQ \equiv \angle MQN + \angle PMQ = \frac{\widehat{MN}}{2} + \frac{\widehat{PQ}}{2} \qquad \blacksquare$$

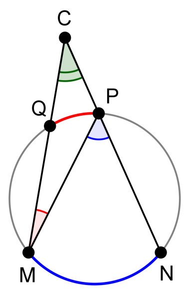
Property 5.9 (Exterior angle in a circle)**.** Let C be a point in the exterior of a circle ω, such that the secants MQ and NP intersect at C. Then, for the *exterior angle* $\angle MCN$, we have:

$$\angle MCN < \frac{\widehat{MN}}{2} \quad \text{and} \quad \angle MCN = \frac{\widehat{MN} - \widehat{PQ}}{2}.$$

Proof. As an exterior angle in $\triangle MPC$, we have

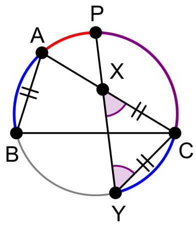
$$\angle MPN = \angle MCP + \angle PMC \equiv \angle MCN + \angle PMQ$$

$$\therefore \frac{\widehat{MN}}{2} = \angle MCN + \frac{\widehat{PQ}}{2}, \text{ i.e. } \angle MCN = \frac{\widehat{MN}}{2} - \frac{\widehat{PQ}}{2} \quad \blacksquare$$



Now, let's see how these two properties can be used in olympiad problems.

Example 5.6 (IGO 2016, Elementary)**.** Let ω be the circumcircle of $\triangle ABC$ with $\overline{AC} > \overline{AB}$. Let X be a point on AC and Y be a point on ω, such that $\overline{CX} = \overline{CY} = \overline{AB}$. (The points A and Y lie on different sides of the line BC). The line XY intersects ω for the second time at P. Show that $\overline{PB} = \overline{PC}$.

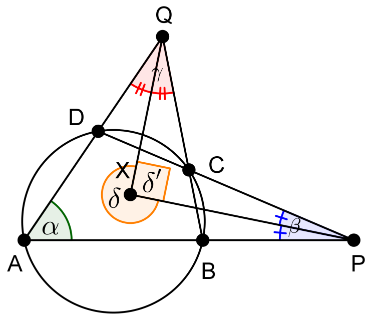


Proof. Triangle $\triangle CXY$ is isosceles, so we have $\angle CXY = \angle CYX$, i.e.

$$\widehat{CP} = 2 \cdot \angle CYP \equiv 2 \cdot \angle CYX = 2 \cdot \angle CXY = \widehat{CY} + \widehat{AP} = \widehat{BA} + \widehat{AP} = \widehat{BP}.$$

Therefore, $\overline{CP} = \overline{BP}$. ∎

Example 5.7. Let $ABCD$ be a cyclic quadrilateral such that the rays AB and DC intersect at P and the rays BC and AD intersect at Q. Prove that the angle bisectors of $\angle BPC$ and $\angle CQD$ are perpendicular.



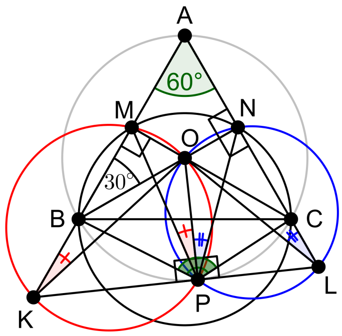
Proof. Let the angle bisectors of $\angle BPC$ and $\angle CQD$ intersect at X. Let's introduce the following notations: $\alpha = \angle BAD$, $\beta = \angle BPC$, $\gamma = \angle CQD$. Then, three of the interior angles in the quadrilateral $APXQ$ are α, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$. Let the fourth interior angle, which is a reflex angle[1], be δ, and let its explementary angle[2] be δ'. We want to prove that $\delta' = 90°$.

From the sum of the interior angles in the quadrilateral $APXQ$, Property 2.4, we get $\alpha + \frac{\beta}{2} + \frac{\gamma}{2} + \delta = 360°$. But also, $\delta + \delta' = 360°$, so $\delta' = \alpha + \frac{\beta}{2} + \frac{\gamma}{2}$. Using the angle measure of arcs, we get $\alpha = \frac{\widehat{BD}}{2} = \frac{\widehat{BC}+\widehat{CD}}{2}$, $\beta = \frac{\widehat{DA}-\widehat{BC}}{2}$ and $\gamma = \frac{\widehat{AB}-\widehat{CD}}{2}$. Finally, we get

$$\begin{aligned}\delta' = \alpha + \frac{\beta}{2} + \frac{\gamma}{2} &= \frac{\widehat{BC}+\widehat{CD}}{2} + \frac{\widehat{DA}-\widehat{BC}}{4} + \frac{\widehat{AB}-\widehat{CD}}{4} = \\ &= \frac{\widehat{AB}+\widehat{BC}+\widehat{CD}+\widehat{DA}}{4} = \frac{360°}{4} = 90°\end{aligned}$$ ∎

[1] A *reflex angle* is an angle that is greater than 180°, but less than 360°.
[2] Two angles are *explementary* if their sum is 360°.

Example 5.8 (IGO 2015, Medium)**.** Let ABC be an equilateral triangle with circumcircle ω and circumcenter O. Let P be a point on the arc $\widehat{BC}$ (that doesn't contain A). The tangent to ω at P intersects lines AB and AC at K and L, respectively. Show that $\angle KOL > 90°$.



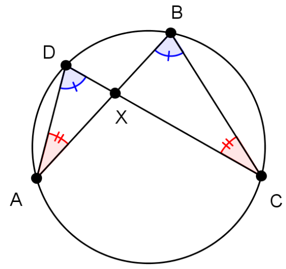
Proof. Similarly as in the previous proof, from the sum of the interior angles in the quadrilateral $AKOL$, we get that $\angle KOL = \angle OLA + \angle LAK + \angle AKO$. But $\angle LAK \equiv \angle CAB = 60°$. So, we need to prove that $\angle OLA + \angle AKO$ is greater than $90° - 60° = 30°$.

In an equilateral triangle, by Property 2.6, the circumcenter is also an orthocenter, so $BO \perp AC$ and $CO \perp AB$. Let M and N be the feet of the altitudes from C and B, respectively. On the other hand, since KL is tangent to (ABC), we get that $OP \perp KL$. Therefore, $\angle OMK + \angle OPK = 90° + 90° = 180°$, so by Property 5.5, the quadrilateral $OMKP$ is cyclic. Therefore, by Property 5.1, $\angle AKO \equiv \angle MKO = \angle MPO$. Similarly, the quadrilateral $ONLP$ is cyclic and therefore $\angle OLA \equiv \angle OLN = \angle OPN$. Therefore, $\angle OLA + \angle AKO$ is equal to $\angle OPN + \angle MPO = \angle MPN$, so we need to prove that $\angle MPN > 30°$.

In the cyclic $ABPC$, by Property 5.5, we get $\angle BPC = 180° - \angle BAC = 120°$. By Thales' Theorem, since $\angle BMC = 90° = \angle BNC$, we know that BC is a diameter in $(BCNM)$. Since $\angle BPC = 120° > 90° = \frac{\widehat{BC}}{2}$, by Property 5.8, we conclude that P must be inside the circle $(BCNM)$. Now, knowing that P is inside this circle, using Property 5.8 again, this time for $\widehat{MN}$, we get $\angle MPN > \frac{\widehat{MN}}{2} = \angle MBN$. Finally, from Sum of angles in triangle ABN, we get $\angle MBN \equiv \angle ABN = 180° - (60° + 90°) = 30°$. Therefore, $\angle MPN > 30°$. ■

Finally, we'll present a few properties concerning lengths in circles.

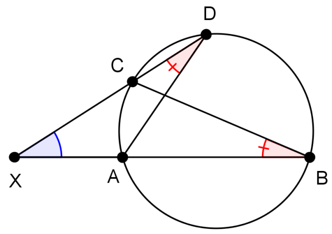
Property 5.10 (Intersecting Chords Theorem)**.** Let AB and CD be two line segments that intersect at X. Then the quadrilateral $ACBD$ is cyclic if and only if $\overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD}$.



Proof. Let's notice that $\angle AXD = \angle CXB$. $\quad (*)$
Then,

$$\begin{aligned} & ACBD \text{ is cyclic} \\ \overset{(5.1)}{\iff} \; & \angle ADC = \angle ABC \quad \text{and} \quad \angle DAB = \angle DCB \\ \iff \; & \angle ADX = \angle XBC \quad \text{and} \quad \angle DAX = \angle XCB \\ \iff \; & \triangle ADX \sim \triangle CBX \\ \overset{(*)}{\iff} \; & \frac{\overline{AX}}{\overline{XD}} = \frac{\overline{CX}}{\overline{XB}} \\ \iff \; & \overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD} \end{aligned}$$ ■

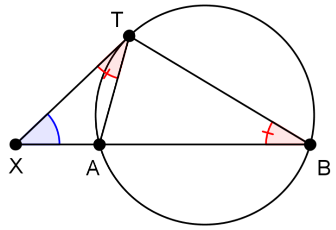
Property 5.11 (Intersecting Secants Theorem)**.** Let AB and CD be two lines that intersect at X, such that $X-A-B$ and $X-C-D$. Then the quadrilateral $ABDC$ is cyclic if and only if $\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD}$.



Proof. Let's notice that $\angle CXB \equiv \angle AXD$. $\quad (*)$
Then,

$$\begin{aligned} & ABDC \text{ is cyclic} \\ \overset{(5.1)}{\iff} \; & \angle ABC = \angle ADC \\ \iff \; & \angle XBC = \angle ADX \\ \overset{(*)}{\iff} \; & \triangle XBC \sim \triangle XDA \\ \overset{(*)}{\iff} \; & \frac{\overline{XB}}{\overline{XC}} = \frac{\overline{XD}}{\overline{XA}} \\ \iff \; & \overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD} \end{aligned}$$ ■

Property 5.12 (Secant-Tangent Theorem)**.** Let ABT be a triangle and let X be a point on AB, such that $X-A-B$. Then XT is tangent to the circumcircle of $\triangle ABT$ if and only if $\overline{XT}^2 = \overline{XA} \cdot \overline{XB}$.



Proof. Let's notice that $\angle TXA \equiv \angle BXT$. $\quad (*)$
Then,

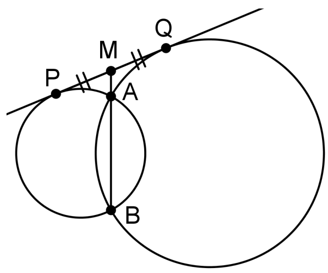
$$\begin{aligned} & XT \text{ is tangent to } (ABT) \\ \overset{(5.3)}{\iff} \; & \angle ATX = \angle TBA \\ \iff \; & \angle ATX = \angle TBX \\ \overset{(*)}{\iff} \; & \triangle XTA \sim \triangle XBT \\ \overset{(*)}{\iff} \; & \frac{\overline{XT}}{\overline{XA}} = \frac{\overline{XB}}{\overline{XT}} \\ \iff \; & \overline{XT}^2 = \overline{XA} \cdot \overline{XB} \end{aligned}$$

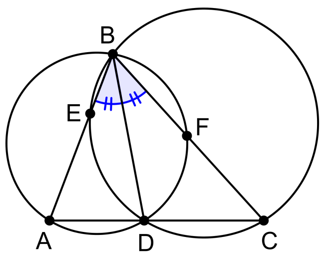
■

Now, let's use these last three properties in Olympiad problems.

Example 5.9. Two circles intersect at A and B. One of their common tangents touches the circles at P and Q. Prove that AB bisects the line segment PQ.

Proof. Let $AB \cap PQ = M$. By Secant-Tangent Theorem for circle (ABP), secant AB and tangent MP, we get $\overline{MP}^2 = \overline{MA} \cdot \overline{MB}$. Similarly, for (ABQ), we get $\overline{MQ}^2 = \overline{MA} \cdot \overline{MB}$. Since $\overline{MP}, \overline{MQ}$ are positive, we get $\overline{MP} = \overline{MQ}$. ■



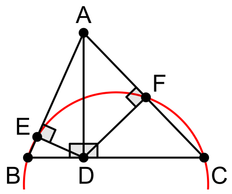


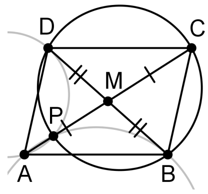
Example 5.10 (St. Petersburg City MO 1996)**.** Let BD be the angle bisector of angle ABC in $\triangle ABC$ with D on the side AC. Let (BDC) meet AB at E and (ABD) meet BC at F. Prove that $\overline{AE} = \overline{CF}$.

Proof. By Intersecting Secants Theorem for circle $(BEDC)$ and secants BE, CD, we get $\overline{AE} \cdot \overline{AB} = \overline{AD} \cdot \overline{AC}$. Similarly, for $(ABFD)$, we get $\overline{CF} \cdot \overline{CB} = \overline{CD} \cdot \overline{CA}$. By dividing these equations side by side, we get $\frac{\overline{AE}}{\overline{CF}} \cdot \frac{\overline{AB}}{\overline{CB}} = \frac{\overline{AD}}{\overline{CD}}$.
From Angle Bisector Theorem, we know that $\frac{\overline{AB}}{\overline{CB}} = \frac{\overline{AD}}{\overline{CD}}$, so $\overline{AE} = \overline{CF}$. ■

Example 5.11. Let AD be an altitude in triangle ABC. Let E and F be the feet of the perpendiculars from D to the sides AB and AC, respectively. Prove that the quadrilateral $BCFE$ is cyclic.

Proof. In the right triangle $\triangle ABD$ with altitude DE, by Euclid's laws, we get $\overline{AD}^2 = \overline{AE} \cdot \overline{AB}$. Similarly, in $\triangle ACD$ we have $\overline{AD}^2 = \overline{AF} \cdot \overline{AC}$. Therefore, $\overline{AE} \cdot \overline{AB} = \overline{AF} \cdot \overline{AC}$, so by the converse of Intersecting Secants Theorem we get that $BCFE$ is cyclic. ∎



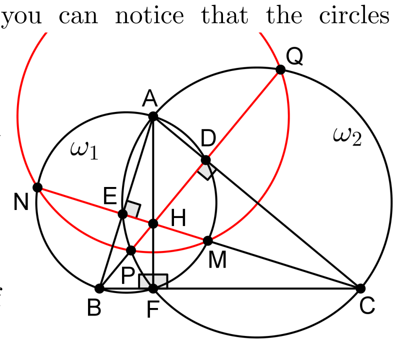


Example 5.12. Let $ABCD$ be a parallelogram with $\overline{AC} > \overline{BD}$. The circumcircle of $\triangle BCD$ intersects AC again at P. Prove that BD is a common tangent for the circumcircles of $\triangle ABP$ and $\triangle ADP$.

Proof. Let M be the intersection of the diagonals AC and BD. From Property 2.16 we know that $\overline{MA} = \overline{MC}$ and $\overline{MB} = \overline{MD}$. Since $BCDP$ is cyclic, by Intersecting Chords Theorem, we get $\overline{MB} \cdot \overline{MD} = \overline{MP} \cdot \overline{MC}$. By substituting the equal lengths, we get $\overline{MB}^2 = \overline{MP} \cdot \overline{MA}$, so by the converse of Secant-Tangent Theorem we get that $MB \equiv BD$ is tangent to (ABP). Similarly, BD is tangent to (ADP). ∎

Example 5.13 (USAMO 1990)**.** Let ABC be an acute-angled triangle. The circle with diameter AB intersects altitude CE and its extension at points M and N, and the circle with diameter AC intersects altitude BD and its extension at points P and Q. Prove that the points M, N, P, Q lie on a common circle.

Proof. If you draw an accurate diagram, you can notice that the circles $\omega_1 \equiv (AMBN)$ and $\omega_2 \equiv (APCQ)$ intersect again on BC. This is not a coincidence; Let's firstly prove this and then try to use this fact. Let ω_1 intersect BC again at F. By Thales' Theorem, we get $\angle AFB = 90°$. Then, $\angle AFC = 180° - \angle AFB = 90°$, so by the converse of Thales' Theorem we get that $F \in \omega_2$, too. Also, since $AF \perp BC$, AF must pass through the orthocenter $H = BD \cap CE$ of $\triangle ABC$.



In order to prove that $MPNQ$ is cyclic, by the converse of Intersecting Chords Theorem, we need to prove that $\overline{MH} \cdot \overline{HN} = \overline{PH} \cdot \overline{HQ}$. But from the Intersecting Chords Theorem we get $\overline{MH} \cdot \overline{HN} \overset{\omega_1}{=} \overline{AH} \cdot \overline{HF} \overset{\omega_2}{=} \overline{PH} \cdot \overline{HQ}$. ∎

Related problems: 29, 30, 31, 32, 33, 34, 35, 39, 41, 43, 44, 45, 46, 47, 48, 49, 50, 52, 55, 56, 57, 58, 59, 60, 62, 66, 67, 71, 72, 73, 74, 75, 79, 81, 86, 87, 94, 98, 99, 123, 125, 136, 142, 145, 150, 157, 169, 170 and 175.

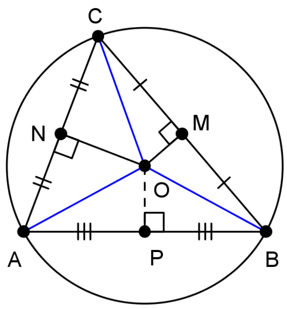
Chapter 6

A Few Important Centers in a Triangle

Property 6.1 (Circumcenter)**.** The three *side bisectors* of a triangle are concurrent. The point of concurrence is the center of a circle that passes through all three vertices of the triangle.

The point of concurrence is called the *circumcenter* of the triangle. The circle that is circumscribed around the triangle is called the *circumcircle* of the triangle.

Proof. Let M, N and P be the midpoints of the sides BC, CA and AB, respectively. Let O be the intersection of the side bisectors of BC and CA. Then $OM \perp BC$ and $ON \perp CA$.



Let's take a look at the triangles $\triangle OMB$ and $\triangle OMC$. They have a common side OM, $\angle OMB = 90° = \angle OMC$ and $\overline{MB} = \overline{MC}$, so by SAS, they are congruent. Therefore, their corresponding sides are equal, i.e. $\overline{OB} = \overline{OC}$. Similarly, $\triangle ONC \cong \triangle ONA$, so $\overline{OC} = \overline{OA}$.

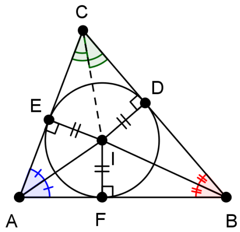
Therefore, $\overline{OA} = \overline{OB}$, so $\triangle OAB$ is isosceles. Therefore, since P is the midpoint of AB, by Property 2.6, we get that OP is the side bisector of AB.

Since $\overline{OA} = \overline{OB} = \overline{OC}$, then O is the center of a circle that passes through the vertices of $\triangle ABC$. ∎

Property 6.2 (Incenter)**.** The three *angle bisectors* of a triangle are concurrent. The point of concurrence is the center of a circle that is tangent to all three sides of the triangle.

The point of concurrence is called the *incenter* of the triangle. The circle that is inscribed inside the triangle is called the *incircle* of the triangle.

Proof. Let I be the intersection of the angle bisectors of $\angle CAB$ and $\angle ABC$. Let D, E and F be the feet of the perpendiculars from I to the sides BC, CA and AB, respectively.



Let's take a look at the triangles $\triangle AIE$ and $\triangle AIF$. They are right triangles and $\angle IAE = \frac{\alpha}{2} = \angle IAF$, so they are similar. But they have a common corresponding side AI, so their ratio of similarity is 1, i.e. they are congruent. Therefore, $\overline{IE} = \overline{IF}$. Similarly, $\triangle BIF \cong \triangle BID$, so $\overline{IF} = \overline{ID}$.

Therefore, $\overline{IE} = \overline{ID}$. The triangles $\triangle CIE$ and $\triangle CID$ are right triangles, so by the Pythagorean Theorem, we get

$$\overline{CE}^2 = \overline{IC}^2 - \overline{IE}^2 = \overline{IC}^2 - \overline{ID}^2 = \overline{CD}^2, \text{ i.e. } \overline{CE} = \overline{CD}.$$

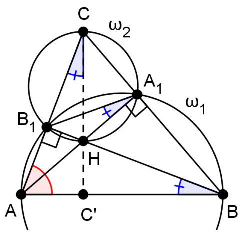
So, by SSS, $\triangle CIE \cong \triangle CID$ and therefore $\angle ICE = \angle ICD$, i.e. CI is the angle bisector of $\angle ECD \equiv \angle ACB$.

Since $\overline{ID} = \overline{IE} = \overline{IF}$, $ID \perp BC$, $IE \perp CA$ and $IF \perp AB$, then I is the center of a circle that is tangent to the sides of $\triangle ABC$. ∎

Property 6.3 (Orthocenter)**.** The three *altitudes* of a triangle are concurrent.

The point of concurrence is called the *orthocenter* of the triangle.

Proof. Let the altitudes AA_1 and BB_1 intersect at H.



Since $AA_1 \perp BC$ and $BB_1 \perp AC$, then $\angle AA_1B = 90^\circ = \angle AB_1B$. Therefore, ABA_1B_1 is a cyclic quadrilateral. Let (ABA_1B_1) be ω_1. Also,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so CB_1HA_1 is a cyclic quadrilateral. Let (CB_1HA_1) be ω_2. Let $CH \cap AB = C'$. Then,

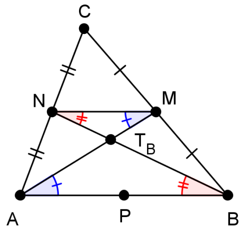
$$\angle ACC' \equiv \angle B_1CH \stackrel{\omega_2}{=} \angle B_1A_1H \equiv \angle B_1A_1A \stackrel{\omega_1}{=} \angle B_1BA \stackrel{\triangle ABB_1}{=} 90^\circ - \alpha$$

Finally, from Sum of angles in triangle $\triangle ACC'$, we get that $\angle AC'C = 90^\circ$, i.e. $CH \perp AB$. ∎

Property 6.4 (Centroid)**.** The three *medians* of a triangle are concurrent. The point of concurrence divides the medians in ratio $2:1$.

The point of concurrence is called the *centroid* of the triangle.

Proof. Let M, N and P be the midpoints of the sides BC, CA and AB, respectively. Then, MN is a midsegment in $\triangle ABC$, so $MN \parallel AB$ and $\overline{AB} = 2 \cdot \overline{MN}$.



Let the B-median intersect the A-median at a point T_B. Then, by Property 2.1, $\angle T_BAB = \angle T_BMN$ and $\angle T_BBA = \angle T_BNM$, so $\triangle T_BAB \sim \triangle T_BMN$ and therefore

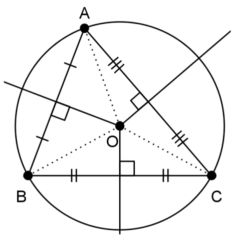
$$\frac{\overline{AT_B}}{\overline{T_BM}} = \frac{\overline{AB}}{\overline{MN}} = 2.$$

Similarly, if the C-median intersect the A-median at T_C, we can get

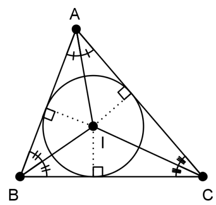
$$\frac{\overline{AT_C}}{\overline{T_CM}} = 2.$$

So $T_B \equiv T_C \equiv T$, i.e. the B-median and the C-median intersect the A-median at the same point T. Additionally,

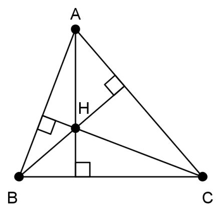
$$\frac{\overline{AT}}{\overline{TM}} = \frac{\overline{BT}}{\overline{TN}} = \frac{\overline{CT}}{\overline{TP}} = 2.$$ ■



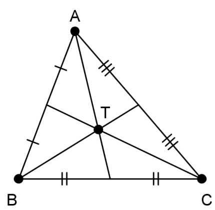
(a) Circumcenter



(b) Incenter



(c) Orthocenter

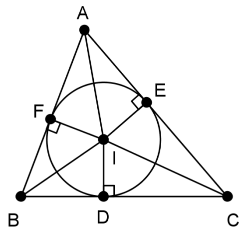


(d) Centroid

Figure 6.1: The four most important centers of a triangle ABC.

Property 6.5. Let s and r be the semiperimeter and the radius of the incircle, respectively, in a triangle $\triangle ABC$. Then,

$$P_{\triangle ABC} = r \cdot s.$$



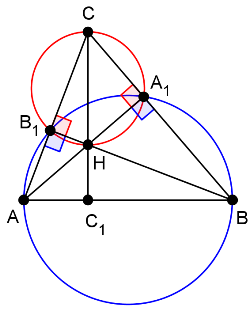
Proof. Let D, E and F be the tangent points of the incircle with the sides BC, CA and AB, respectively. Let I be the incenter of $\triangle ABC$. Then $ID \perp BC$, $IE \perp CA$ and $IF \perp AB$.

$$\begin{aligned} P_{\triangle ABC} &= P_{\triangle BCI} + P_{\triangle CAI} + P_{\triangle ABI} = \\ &= \frac{\overline{BC} \cdot \overline{ID}}{2} + \frac{\overline{CA} \cdot \overline{IE}}{2} + \frac{\overline{AB} \cdot \overline{IF}}{2} = \\ &= \frac{a \cdot r}{2} + \frac{b \cdot r}{2} + \frac{c \cdot r}{2} = \\ &= r \cdot \frac{a+b+c}{2} = \\ &= r \cdot s \end{aligned}$$

■

Property 6.6. Let AA_1, BB_1 and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Then,

- ABA_1B_1, BCB_1C_1 and CAC_1A_1 are cyclic quadrilaterals
- AB_1HC_1, BC_1HA_1 and CA_1HB_1 are cyclic quadrilaterals



Proof. Since $AA_1 \perp BC$ and $BB_1 \perp AC$, then $\angle AA_1B = 90^\circ = \angle AB_1B$. Therefore, ABA_1B_1 is a cyclic quadrilateral. Similarly, BCB_1C_1 and CAC_1A_1 are cyclic quadrilaterals. □

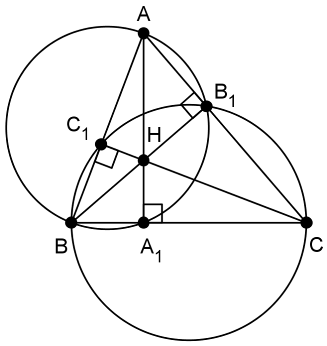
On the other hand,

$$\angle CB_1H + \angle CA_1H \equiv \angle CB_1B + \angle CA_1A = 90^\circ + 90^\circ = 180^\circ,$$

so CA_1HB_1 is a cyclic quadrilateral. Similarly, AB_1HC_1 and BC_1HA_1 are cyclic quadrilaterals. ■

Property 6.7. Let AA_1, BB_1 and CC_1 be the altitudes in a $\triangle ABC$ and let H be its orthocenter. Then,

$$\overline{AH} \cdot \overline{HA_1} = \overline{BH} \cdot \overline{HB_1} = \overline{CH} \cdot \overline{HC_1}.$$



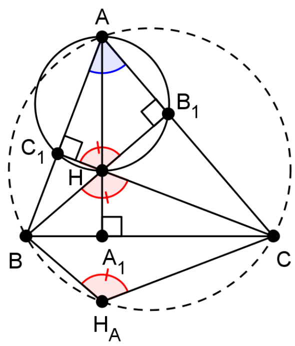
Proof. From Property 6.6, we know that ABA_1B_1 is a cyclic quadrilateral. Since the altitudes AA_1 and BB_1 intersect at the orthocenter H, by the Intersecting Chords Theorem, we get

$$\overline{AH} \cdot \overline{HA_1} = \overline{BH} \cdot \overline{HB_1}.$$

Similarly, from the cyclic quadrilateral BCB_1C_1, we get

$$\overline{BH} \cdot \overline{HB_1} = \overline{CH} \cdot \overline{HC_1}.$$ ■

Property 6.8. The reflections of the orthocenter with respect to the sides of a triangle lie on the circumcircle of the triangle.

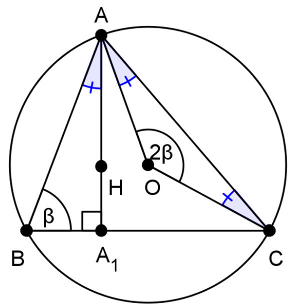


Proof. Let B_1 and C_1 be the feet of the altitudes from B and C, respectively, in $\triangle ABC$. Let H be its orthocenter. From Property 6.6, we know that AB_1HC_1 is a cyclic quadrilateral and therefore $\angle B_1HC_1 = 180° - \alpha$. As vertical angles, $\angle BHC = \angle B_1HC_1$. Let H_A be the reflection of H with respect to the side BC. By symmetry, $\angle BH_AC = \angle BHC$. Therefore, $\angle BH_AC = 180° - \alpha$. Finally,

$$\angle CAB + \angle BH_AC = \alpha + 180° - \alpha = 180°,$$

so $H_A \in (ABC)$. Similarly, $H_B, H_C \in (ABC)$. ■

Property 6.9. The orthocenter and the circumcenter in a triangle are isogonal conjugates[1].



Proof. WLOG, $\overline{AB} < \overline{AC}$. Let O and H be the circumcenter and the orthocenter, respectively, in $\triangle ABC$. We need to prove that $\angle HAB = \angle OAC$.

Let A_1 be the foot of the altitude from A to BC. Then, from $\triangle ABA_1$, we get

$$\angle HAB \equiv \angle A_1AB = 90^\circ - \angle ABA_1 = 90^\circ - \beta. \tag{1}$$

Since $\angle ABC$ and $\angle AOC$ are inscibed and central angle, respectively, over $\widehat{AC}$ in (ABC), we have

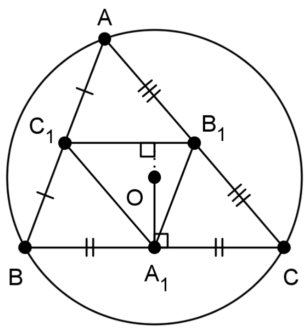
$$\angle AOC = 2 \cdot \angle ABC = 2\beta.$$

Since $\overline{OA} = R = \overline{OC}$, from sum of the angles in the isosceles $\triangle AOC$, we have

$$\angle OAC = \frac{180^\circ - \angle AOC}{2} = \frac{180^\circ - 2\beta}{2} = 90^\circ - \beta. \tag{2}$$

From (1) and (2), we get that $\angle HAB = \angle OAC$. Similarly, $\angle HBC = \angle OBA$ and $\angle HCA = \angle OCB$, so H and O are isogonal conjugates in $\triangle ABC$. ■

Property 6.10. The circumcenter of a triangle is the orthocenter of its medial triangle[2].

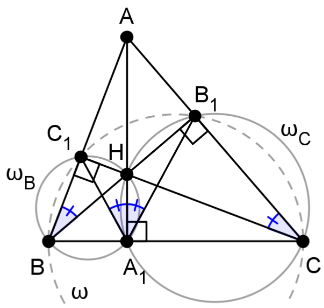


Proof. Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB in $\triangle ABC$, respectively. Let O be the circumcenter of $\triangle ABC$. Since A_1 is the midpoint of the chord BC in (ABC), $OA_1 \perp BC$. Since B_1C_1 is the midsegment in $\triangle ABC$, $B_1C_1 \parallel BC$. Therefore, $OA_1 \perp B_1C_1$, i.e. A_1O is an altitude in $\triangle A_1B_1C_1$. Similarly, B_1O and C_1O are also altitudes in $\triangle A_1B_1C_1$, so O is the orthocenter of $\triangle A_1B_1C_1$. ■

[1] Two points P and P^* are called isogonal conjugates if XP^* is the reflection of XP across the angle bisector of the angle at the vertex X in a triangle $\triangle ABC$, where X is any of the vertices A, B or C. In other words, the lines XP and XP^* make equal angles with the sides of the triangle that contain X, e.g. $\angle PAB = \angle P^*AC$.

[2] The medial triangle is the triangle with vertices the midpoints of a triangle.

Property 6.11. The orthocenter of a triangle is the incenter of its orthic triangle[3].



Proof. Let AA_1, BB_1 and CC_1 be the altitudes in a $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$. We want to prove that A_1H is the angle bisector of $\angle C_1A_1B_1$, i.e. $\angle C_1A_1H = \angle HA_1B_1$. From Property 6.6, we know that BC_1HA_1, CA_1HB_1 and BCB_1C_1 are cyclic quadrilaterals. Let's call them ω_B, ω_C and ω, respectively. Then,

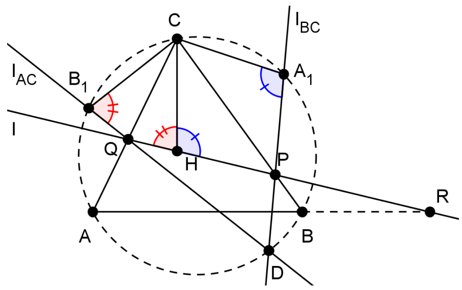
$$\angle C_1A_1H \stackrel{\omega_B}{=} \angle C_1BH \equiv \angle C_1BB_1 \stackrel{\omega}{=} \angle C_1CB_1 \equiv \angle HCB_1 \stackrel{\omega_C}{=} \angle HA_1B_1.$$

Similarly, B_1H and C_1H are angle bisectors of $\angle A_1B_1C_1$ and $\angle B_1C_1A_1$, so H is the incenter of $\triangle A_1B_1C_1$. ∎

Property 6.12. Let l be any line through the orthocenter of $\triangle ABC$. Prove that the reflections of the line l with respect to the lines AB, BC and CA are concurrent at the circumcircle of $\triangle ABC$.

Proof. Let the line l intersect the lines BC, CA and AB at P, Q and R, respectively. We will examine the case when H is inside $\triangle ABC$ (the other cases should be similar). Since one of these points will be on the extension of a side and two of these points will be on the sides of the triangle, WLOG, let R be on the extension of the side AB.

Let A_1, B_1 and C_1 be the reflections of the orthocenter with respect to the sides BC, CA and AB, respectively. From Property 6.8, we know that $A_1, B_1, C_1 \in (ABC)$. Therefore, l_{BC}, the reflection of the line l with respect to the line BC, will contain A_1 (and similarly for the other lines). Let D be the intersection of the lines l_{BC} and l_{AC}. We want to prove that $D \in (ABC)$.



$$\angle DA_1C + \angle CB_1D \equiv \angle PA_1C + \angle CB_1Q = \angle PHC + \angle CHQ = 180°$$

$$\therefore D \in (A_1CB_1) \equiv (ABC)$$

Similarly, we can prove that the intersection of the lines l_{AB} and l_{BC} lies on (ABC). ∎

Related problems: 7, 14, 16, 17, 19, 63, 64, 65, 68, 70 and 76.

[3]The orthic triangle is the triangle with vertices the feet of the altitudes of a triangle.

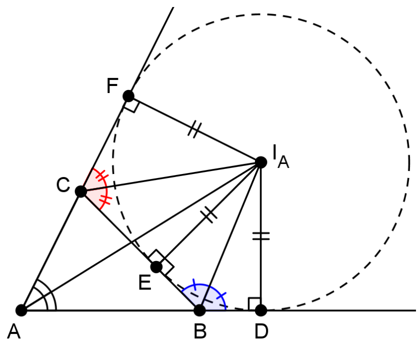
Chapter 7

Excircles

Property 7.1 (Excenter)**.** The external bisectors of two of the angles and the internal angle bisector of the third angle in a triangle are concurrent. The point of concurrence is the center of a circle that is externally tangent to one of the sides and the extensions of the other two sides of a triangle.

The point of concurrence is called an *excenter* of the triangle. The circle that is exscribed outside the triangle is called an *excircle* of the triangle. There are three excircles for each triangle.

Proof. Let I_A be the intersection of the external angle bisectors at B and C, in a triangle $\triangle ABC$. Let D, E, and F be the feet of the perpendiculars from I_A to the lines AB, BC and AC, respectively. The triangles $\triangle BDI_A$ and $\triangle BEI_A$



are similar because they have two equal angles. Moreover, they have a common corresponding side, so they are congruent. Therefore, $\overline{I_AD} = \overline{I_AE}$. Similarly, $\overline{I_AF} = \overline{I_AE}$. The triangles $\triangle I_ADA$ and $\triangle I_AFA$ are right triangles with two equal corresponding sides, so by the Pythagorean Theorem, the third sides are also equal. By the criterion SSS, these triangles are congruent. Therefore, their corresponding angles $\angle I_AAD$ and $\angle I_AAF$ are equal, so AI_A is angle bisector of $\angle BAC$.

Since $\overline{I_AD} = \overline{I_AE} = \overline{I_AF}$, $I_AD \perp AB$, $I_AE \perp BC$ and $I_AF \perp CA$, then I is the center of a circle that is tangent to the lines AB, BC and CA. ■

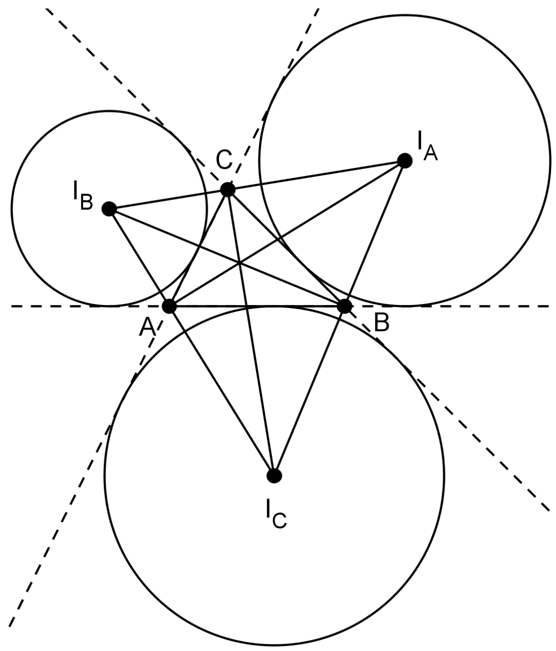
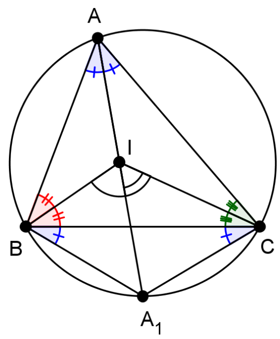


Figure 7.1: The three excircles of a triangle ABC.

Property 7.2. Let I be the incenter of $\triangle ABC$. Let A_1 be the second intersection of the angle bisector of $\angle BAC$ with the circumcircle of $\triangle ABC$. Prove that
$$\overline{A_1B} = \overline{A_1I} = \overline{A_1C}.$$



Proof.

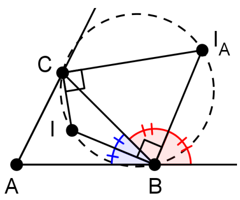
$$\angle A_1BI = \angle A_1BC + \angle CBI = \angle A_1AC + \frac{\beta}{2} = \frac{\alpha}{2} + \frac{\beta}{2}$$

$$\angle A_1IB = \angle IAB + \angle IBA = \frac{\alpha}{2} + \frac{\beta}{2}$$

$$\therefore \triangle A_1BI \text{ is isosceles, i.e. } \overline{A_1B} = \overline{A_1I}$$

Similarly, $\overline{A_1C} = \overline{A_1I}$. ■

Property 7.3. Let I and I_A be the incenter and A-excenter in $\triangle ABC$, respectively. Prove that the quadrilateral IBI_AC is cyclic.

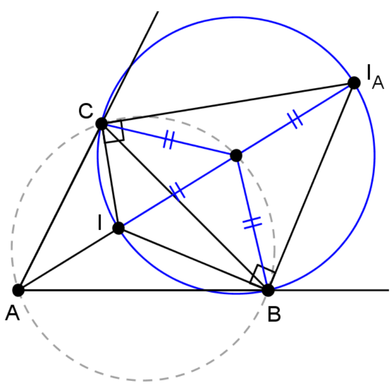


Proof. Since I and I_A are incenter and excenter, BI and BI_A are internal and external angle bisectors.

$$\therefore \angle IBI_A = \angle IBC + \angle CBI_A = \frac{\beta}{2} + \frac{180^\circ - \beta}{2} = 90^\circ$$

Similarly, $\angle ICI_A = 90^\circ$.
Therefore, $\angle IBI_A + \angle ICI_A = 180^\circ$, so IBI_AC is cyclic. ■

With these two properties, we actually proved that the circle with diameter II_A passes through B and C and its center is the intersection of AI with the circumcircle of $\triangle ABC$.

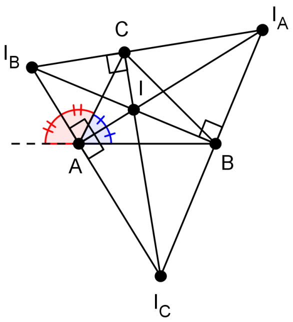


Property 7.4. Let I_A, I_B and I_C be the excenters opposite of A, B, and C in $\triangle ABC$, respectively. Prove that the incenter of $\triangle ABC$ is the orthocenter of $\triangle I_A I_B I_C$.

Proof. Let I be the incenter of $\triangle ABC$. Since I and I_B are incenter and excenter, AI and AI_B are internal and external angle bisectors.

$$\therefore \angle IAI_B = \angle IAC + \angle CAI_B = \frac{\alpha}{2} + \frac{180° - \alpha}{2} = 90°$$

Similarly, $\angle IAI_C = 90°$. Therefore, since $\angle IAI_B + \angle IAI_C = 180°$, $A \in I_B I_C$ and $I_A A \equiv IA \perp I_B I_C$, so $I_A A$ is an altitude in $\triangle I_A I_B I_C$. Similarly, $I_B B$ and $I_C C$ are altitudes, too, so I is the orthocenter of $\triangle I_A I_B I_C$. ∎



Property 7.5. Let I and I_A be the incenter and the A-excenter in $\triangle ABC$. Prove that

$$\overline{AI} \cdot \overline{AI_A} = \overline{AB} \cdot \overline{AC}.$$

Proof 1. Let's look at the triangles $\triangle AIC$ and $\triangle ABI_A$.

$$\angle CAI = \frac{\alpha}{2} = \angle I_A AB \tag{1}$$

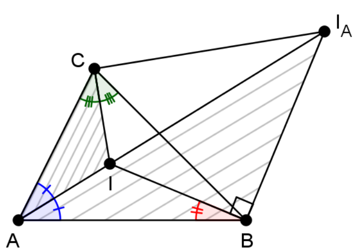
$$\angle AIC = 180° - (\angle IAC + \angle ICA) = 180° - \left(\frac{\alpha + \gamma}{2}\right) = 90° + \frac{\beta}{2}$$

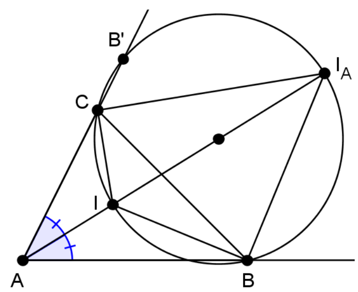
$$\angle ABI_A = \angle ABI + \angle IBI_A = \frac{\beta}{2} + 90°$$

$$\therefore \angle AIC = \angle ABI_A \tag{2}$$

From (1) and (2), we can conclude that $\triangle AIC \sim \triangle ABI_A$.

$$\therefore \frac{\overline{AI}}{\overline{AC}} = \frac{\overline{AB}}{\overline{AI_A}}, \text{ i.e. } \overline{AI} \cdot \overline{AI_A} = \overline{AB} \cdot \overline{AC}$$ ∎





Proof 2. Recall from Property 7.3 that IBI_AC is a cyclic quadrilateral and that the center of this circle lies on AI. Notice that the line AC is a reflection of the line AB with respect to the angle bisector of $\angle BAC$, AI. Let the second intersection of (IBI_AC) with AC be B'. By symmetry, $\overline{AB} = \overline{AB'}$. Now, by the Intersecting Secants Theorem for the point A, we have

$$\overline{AI} \cdot \overline{AI_A} = \overline{AB'} \cdot \overline{AC} = \overline{AB} \cdot \overline{AC}$$ ∎

Related problems: 53, 77, 80, 85, 93, 115, 128 and 160.

Chapter 8

Collinearity

Three points are *collinear* if they lie on a single line. We will now present a few approaches that will help us prove that three points are collinear when solving geometry problems.

8.1 Manual Approach

There are three most common angle chasing ways to prove that three points A, B and C are collinear.

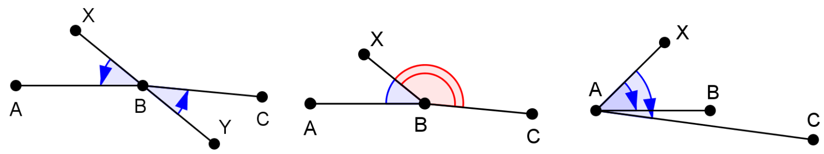


Figure 8.1: Three collinearity configurations

In the first configuration, we will need two extra points that are already collinear with our "middle" point B. Let those points be X and Y. If $\angle XBA = \angle YBC$, then the points A, B and C are collinear.

In the second configuration, we will need one extra point X that doesn't lie on the supposed line $A - B - C$. If $\angle ABX + \angle XBC = 180°$, then the points A, B and C are collinear.

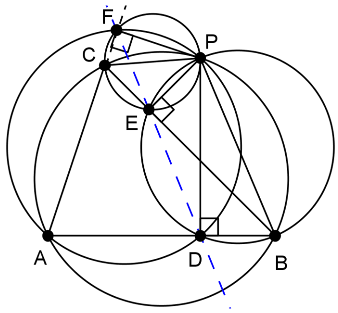
In the third configuration, we will also need one extra point X that doesn't lie on the supposed line $A - B - C$. If $\angle XAB = \angle XAC$, then the points A, B and C are collinear.

In the proof of the following theorem, we will demonstrate all three approaches.

Example 8.1 (Simson Line Theorem)**.** Let P be a point on the circumcircle ω of a triangle ABC. If D, E and F are the feet of the perpendiculars from P to the lines AB, BC and CA, prove that the points D, E and F are collinear.

Proof. WLOG let P be on the arc $\widehat{BC}$ that doesn't contain A.

$$\begin{aligned}
&\angle PDB = 90^\circ = \angle PEB \\
&\quad \therefore PEDB \text{ is cyclic} && (1) \\
&\angle CEP + \angle CFP = 180^\circ \\
&\quad \therefore CEPF \text{ is cyclic} && (2) \\
&\angle ADP + \angle AFP = 180^\circ \\
&\quad \therefore ADPF \text{ is cyclic} && (3)
\end{aligned}$$



We will now finish the proof in three different ways, demonstrating all of the approaches mentioned before.
I way: We will prove that $\angle CEF = \angle BED$.

$$\angle CEF \overset{(2)}{=} \angle CPF \overset{\triangle CFP}{=} 90^\circ - \angle FCP$$

$$\angle BED \overset{(1)}{=} \angle BPD \overset{\triangle BDP}{=} 90^\circ - \angle DBP$$

$$\angle FCP = 180^\circ - \angle ACP \overset{\omega}{=} \angle ABP \equiv \angle DBP$$

II way: We will prove that $\angle FEP + \angle PED = 180^\circ$.

$$\angle FEP \overset{(2)}{=} \angle FCP = 180^\circ - \angle PCA$$

$$\angle PED \overset{(1)}{=} 180 - \angle PBD \equiv 180^\circ - \angle PBA$$

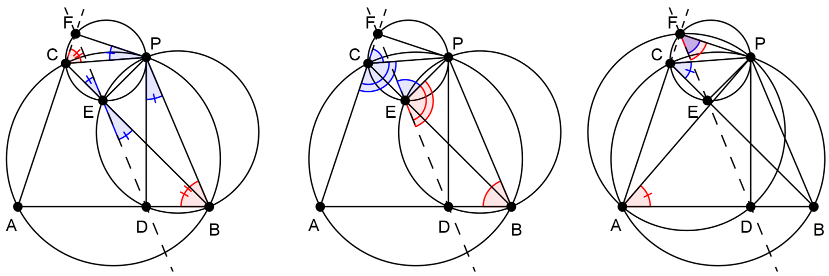
$$\angle FEP + \angle PED = 360^\circ - (\angle PCA + \angle PBA) \overset{\omega}{=} 360^\circ - 180^\circ = 180^\circ$$

III way: We will prove that $\angle PFE = \angle PFD$.

$$\angle PFE \overset{(2)}{=} \angle PCE \equiv \angle PCB$$

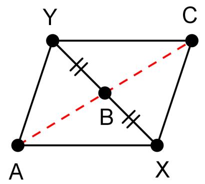
$$\angle PFD \overset{(3)}{=} \angle PAD \equiv \angle PAB$$

$$\angle PCB \overset{\omega}{=} \angle PAB \qquad \blacksquare$$



8.2 Parallelogram Trick

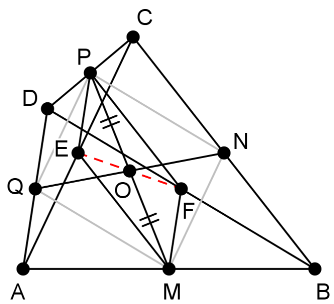
Sometimes (although much more rarely), we can use the following approach to prove that three points A, B and C are collinear. If we know that the "middle" point B is the midpoint of some line segment XY, then by showing that $AXCY$ is a parallelogram, we will prove that A, B and C are collinear. This is because we know that the diagonals in a parallelogram bisect at the intersection point, so if B is the midpoint of the diagonal XY, then it must also be the midpoint of the other diagonal AC, i.e. it must lie on AC.



We will now solve one problem as an example of how this approach can be used.

Example 8.2. Prove that in any convex quadrilateral $ABCD$ the midpoints of its diagonals and the point which is the intersection of the lines through the midpoints of the opposite sides are collinear.

Proof. Let E and F be the midpoints of the diagonals AC and BD, respectively. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively, and let O be the intersection of MP and NQ. We need to prove that E, O and F are collinear.



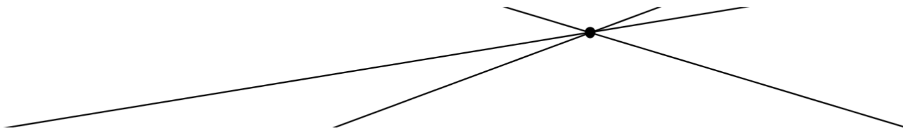
Firstly, let's take a look at the quadrilateral $MNPQ$. MN is midsegment in $\triangle ABC$. Therefore, $MN \parallel AC$ and $\overline{MN} = \frac{\overline{AC}}{2}$. Similarly, PQ is midsegment in $\triangle DAC$, so $PQ \parallel AC$ and $\overline{PQ} = \frac{\overline{AC}}{2}$. Therefore, $MN \parallel PQ$ and $\overline{MN} = \overline{PQ}$. Thus, by Property 2.19, $MNPQ$ is a parallelogram. Since we know that the diagonals in a parallelogram bisect at the intersection point and O is the intersection of the diagonals MP and NQ, we get that O is the midpoint of MP.

Now, since we want to prove that E, O and F are collinear and we know that O is the midpoint of MP, it is enough to prove that $EMFP$ is a parallelogram. Notice that ME is midsegment in $\triangle ABC$. Therefore, $ME \parallel BC$ and $\overline{ME} = \frac{\overline{BC}}{2}$. Similarly, FP is midsegment in $\triangle BCD$, so $FP \parallel BC$ and $\overline{FP} = \frac{\overline{BC}}{2}$. Therefore, $ME \parallel FP$ and $\overline{ME} = \overline{FP}$. Thus, by Property 2.19, $EMFP$ is a parallelogram. ■

Related problems: (Collinearity) 36, 37, 38, 40, 42, 49, 84 and 143.

Chapter 9

Concurrence

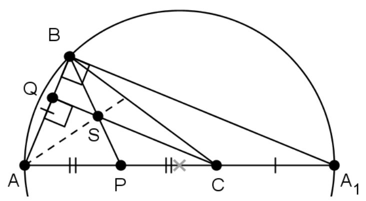


Three lines are *concurrent* if they pass through a common point. We will firstly present a few approaches to proving concurrence by using things we have already visited during our journey. Then, we will learn a new theorem related to concurrence.

9.1 Manual Approach

The most basic way to prove that three lines are concurrent is to take the intersection of two of them and then somehow prove that the third line passes through this intersection.

Example 9.1. Let C be a point on the diameter AA_1 in a circle ω. Let B be a point on ω, such that $\overline{AB} = \overline{CA_1}$. Prove that in $\triangle ABC$, the internal angle bisector at the vertex A, the median from the vertex B and the altitude from the vertex C are concurrent.



Proof. We will take the intersection of the median and the altitude and we will prove that the angle bisector passes through this point. Let P be the midpoint of AC and let Q be the foot of the altitude from the vertex C in $\triangle ABC$. Let $S = BP \cap CQ$. We need to prove that AS bisects the angle $\angle CAB$.

$$CQ \perp AB \quad (\because CQ \text{ is altitude in } \triangle ABC)$$

$$A_1B \perp AB \quad (\because AA_1 \text{ is diameter})$$

$$\therefore CQ \parallel A_1B \text{ , i.e. } CS \parallel A_1B$$

$$\therefore \frac{\overline{PC}}{\overline{CA_1}} = \frac{\overline{PS}}{\overline{SB}} \text{ (by Thales' Proportionality Theorem)}$$

Substituting $\overline{PC} = \overline{AP}$ and $\overline{CA_1} = \overline{AB}$, we get

$$\frac{\overline{AP}}{\overline{AB}} = \frac{\overline{PS}}{\overline{SB}},$$

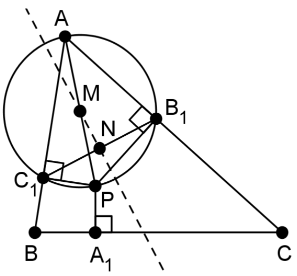
which by the Angle Bisector Theorem means that AS is the angle bisector of $\angle PAB$. Since $\angle PAB \equiv \angle CAB$, we get that AS bisects $\angle CAB$. ■

Remark. This approach can, in fact, be used not only for proving concurrent lines, but also for proving that any three objects (lines or circles or any combination of those) pass through a common point. For example, if we need to prove that three circles pass through a point, we will take the intersection of two of the circles and then prove that this intersection lies on the third circle. Otherwise, if we need to prove that two lines intersect on a circle, we can either take the intersection of the lines and prove that this intersection lies on the circle, or we can take the intersection of one of the lines and the circle and prove that this intersection lies on the other line.

9.2 Special Lines

Another way to prove that three lines are concurrent is by proving that they are "special lines" (such as side bisectors, angle bisectors, altitudes, ...) in a triangle in the figure. This is because we already know that these special lines concur at one of the important centers that we mentioned in chapter 6.

Example 9.2. Let P be an arbitrary point inside the triangle ABC. Let A_1, B_1 and C_1 be the feet of the perpendiculars from P to BC, CA and AB, respectively. Prove that the lines that pass through the midpoints of PA and B_1C_1, PB and C_1A_1, and PC and A_1B_1 are concurrent.



Proof. We will prove that these lines are in fact side bisectors in $\triangle A_1B_1C_1$, so they will concur at the circumcenter of $\triangle A_1B_1C_1$. Let M and N be the midpoints of PA and B_1C_1, respectively.

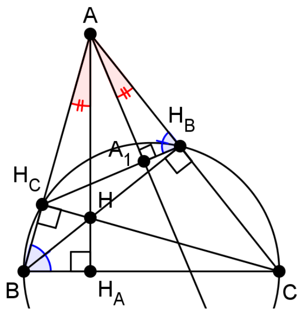
$$\angle PC_1A + \angle PB_1A = 180° \quad (\because PC_1 \perp AB,\ PB_1 \perp AC)$$

Therefore, AC_1PB_1 is cyclic. Since $\angle AC_1P = 90°$, AP is a diameter in (AC_1PB_1), so M is its center. Since N is the midpoint of the chord B_1C_1 and M is the center, MN is the side bisector of B_1C_1. Similarly, the other two lines are also side bisectors in $\triangle A_1B_1C_1$, so they are concurrent. ■

9.3 Special Point

If the lines in question are not "special lines", there is another way that the important centers can help us–by proving somehow that the lines pass through a "special point", i.e. an important center in a triangle in the figure.

Example 9.3 (Macedonia MO 2015)**.** Let AH_A, BH_B and CH_C be altitudes in $\triangle ABC$. Let p_A, p_B and p_C be the perpendicular lines from vertices A, B and C to H_BH_C, H_CH_A and H_AH_B, respectively. Prove that p_A, p_B and p_C are concurrent.



Proof. We will prove that the lines pass through the circumcenter of $\triangle ABC$. Let $A_1 = p_A \cap H_BH_C$.

$$BCH_BH_C \text{ is cyclic } (\because \angle BH_BC = 90^\circ = \angle BH_CC)$$

$$\therefore \angle AH_BA_1 \equiv \angle AH_BH_C = \angle H_CBC \equiv \angle ABC = \beta$$

$$\angle CAA_1 \equiv \angle H_BAA_1 = 90^\circ - \angle AH_BA_1 = 90^\circ - \beta \quad (\because AA_1 \perp H_BH_C)$$

$$\angle BAH_A = 90^\circ - \angle ABH_A = 90^\circ - \beta$$

In conclusion, $\angle CAA_1 = \angle BAH_A$, so AH_A and $AA_1 \equiv p_A$ are symmetric with respect to the angle bisector of $\angle BAC$. Since the orthocenter lies on the altitude AH_A, its isogonal conjugate, the circumcenter (Property 6.9), must lie on p_A. Similarly, the circumcenter of $\triangle ABC$ lies on p_B and p_C, so the lines p_A, p_B and p_C are concurrent. ∎

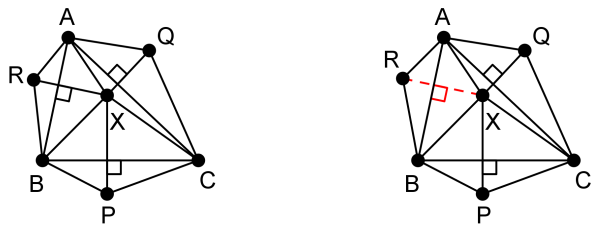
9.4 Concurrent Perpendiculars

Property 9.1 (Carnot's Extended Theorem)**.** Let P, Q and R be points in the plane of triangle ABC. Then, the lines ℓ_P, ℓ_Q and ℓ_R, which are the perpendiculars from P, Q and R to BC, CA and AB, respectively, are concurrent if and only if

$$\overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$$

Proof. Let's prove the first direction, i.e. let ℓ_P, ℓ_Q and ℓ_R be concurrent and let the point of concurrence be X. By the perpendicularity condition in Property 4.3, we get that $XP \perp BC \iff \overline{PB}^2 - \overline{PC}^2 = \overline{XB}^2 - \overline{XC}^2$. If we substitute this for all three perpendiculars, we get

$$\overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 =$$
$$= \overline{XB}^2 - \overline{XC}^2 + \overline{XC}^2 - \overline{XA}^2 + \overline{XA}^2 - \overline{XB}^2 = 0 \quad \square$$

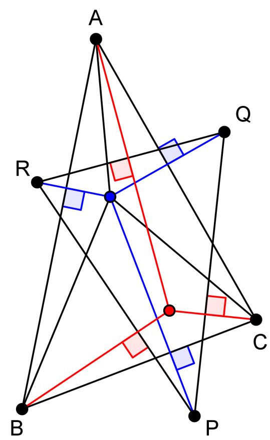


Now, for the other direction, let $\overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$. Let $\ell_P \cap \ell_Q = X$. In order for the three perpendiculars to be concurrent, we need to prove that $X \in \ell_R$.

By using the same perpendicularity condition as before for the perpendiculars ℓ_P and ℓ_Q and substituting in the given condition, we get

$$\overline{XB}^2 - \overline{XC}^2 + \overline{XC}^2 - \overline{XA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$$
$$\overline{XB}^2 - \overline{XA}^2 = \overline{RB}^2 - \overline{RA}^2$$
$$\therefore XR \perp AB \quad \text{(by Property 4.3)}$$
$$\therefore X \in \ell_R \qquad \blacksquare$$

Property 9.2. Let P, Q and R be points in the plane of triangle ABC. Then, the perpendiculars from P, Q and R to BC, CA and AB, respectively, are concurrent if and only if the perpendiculars from C, A and B to PQ, QR and RP, respectively, are concurrent.



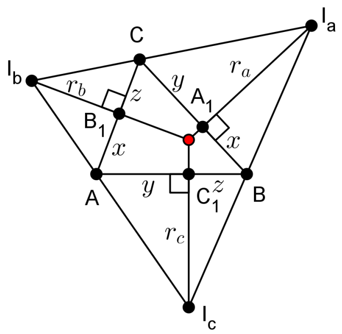
Proof. By using Carnot's Extended Theorem, rearranging the terms and using Carnot's Extended Theorem again, we get

$$LHS \iff$$
$$\iff \overline{PB}^2 - \overline{PC}^2 + \overline{QC}^2 - \overline{QA}^2 + \overline{RA}^2 - \overline{RB}^2 = 0$$
$$\iff 0 = \overline{CP}^2 - \overline{CQ}^2 + \overline{AQ}^2 - \overline{AR}^2 + \overline{BR}^2 - \overline{BP}^2$$
$$\iff RHS \qquad \blacksquare$$

In the following problem, we will present a solution with each of the aforementioned properties.

Example 9.4 (Serbia 2017, Drzavno IIIA)**.** Let I_a, I_b and I_c be the excenters of triangle ABC opposite the vertices A, B and C, respectively. Let A_1, B_1 and C_1 be the tangent points of the $A-$, $B-$ and $C-$excircle with the sides BC, CA and AB, respectively. Prove that the lines I_aA_1, I_bB_1 and I_cC_1 are concurrent.

Proof 1. By Carnot's Extended Theorem the three perpendiculars are concurrent if and only if $\overline{I_aB}^2 - \overline{I_aC}^2 + \overline{I_bC}^2 - \overline{I_bA}^2 + \overline{I_cA}^2 - \overline{I_cB}^2 = 0$

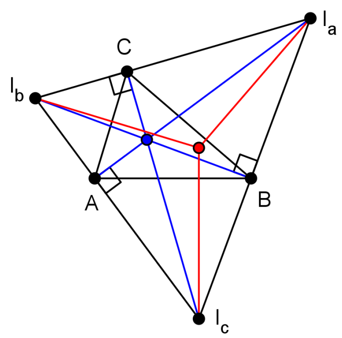


As we will shortly see (in Property 10.1.3), the tangent points of the excircles have such property that $\overline{BA_1} = s - c = \overline{AB_1}$, where s is the semiperimeter of $\triangle ABC$. Let $x = s - c$, $y = s - b$ and $z = s - a$ and let r_a, r_b and r_c be the radii of the $A-$, $B-$ and $C-$excircle, respectively. By using the Pythagorean Theorem six times, the above statement is equivalent to

$$r_a^2 + x^2 - r_a^2 - y^2 + r_b^2 + z^2 - r_b^2 - x^2 + r_c^2 + y^2 - r_c^2 - z^2 = 0$$

which is true because everything on the left-hand side cancels out. ■

Proof 2. By Property 9.2, the perpendiculars from I_a, I_b and I_c to BC, CA and AB, respectively, are concurrent if and only if the perpendiculars from C, A and B to I_aI_b, I_bI_c and I_cI_a, respectively, are concurrent. We are going to prove the latter.



Let's recall, from Property 7.4, that A, B and C are the feet of the altitudes in $\triangle I_aI_bI_c$. Thus, the perpendiculars from C, A and B to I_aI_b, I_bI_c and I_cI_a are in fact the altitudes in $\triangle I_aI_bI_c$, so they concur at its orthocenter. ■

Related problems: (Concurrence) 17, 101 and 148.

Chapter 10

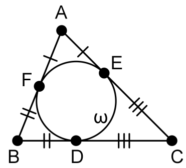
A Few Useful Lemmas

10.1 Tangent Segments

From chapter 5, we know that the tangent segments from a point to the circle are of equal length. We will now present some useful properties based on this fact.

Property 10.1.1. Let ω be the incircle in $\triangle ABC$. Let D be the tangent point of ω with the side BC. Prove that $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$.

Proof. Let E and F be the tangent points of ω with the sides CA and AB, respectively. Then, as tangent segments from A, B and C to ω, we get

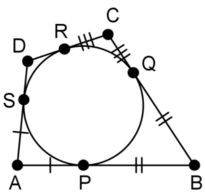


$$\overline{AF} = \overline{AE}\,, \quad \overline{BF} = \overline{BD} \quad \text{and} \quad \overline{CD} = \overline{CE}.$$

$\therefore \overline{AB}+\overline{CD} = \overline{AF}+\overline{FB}+\overline{CD} = \overline{AE}+\overline{EC}+\overline{BD} = \overline{AC}+\overline{BD}$ ■

Property 10.1.2 (Tangential quadrilateral)**.** Let $ABCD$ be a quadrilateral such that there exists an incircle that is tangent to its sides. Prove that the sums of the opposite sides are equal, i.e.

$$\overline{AB} + \overline{CD} = \overline{BC} + \overline{AD}.$$



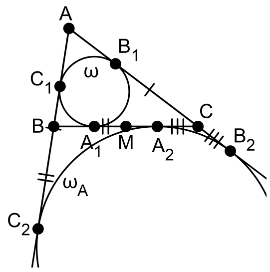
Proof. Let P, Q, R and S be the tangent points of the incircle with the sides AB, BC, CD and DA, respectively. Then, as tangent segments,

$$\overline{AP} = \overline{AS}\,, \quad \overline{BP} = \overline{BQ}\,, \quad \overline{CQ} = \overline{CR} \quad \text{and} \quad \overline{DR} = \overline{DS}.$$

$\therefore \overline{AB}+\overline{CD} = \overline{AP}+\overline{PB}+\overline{CR}+\overline{RD} = \overline{AS}+\overline{BQ}+\overline{CQ}+\overline{DS} = \overline{BC}+\overline{AD}$ ■

Property 10.1.3. Let ω and ω_A be the incircle and the A-excircle in $\triangle ABC$. Let A_1, B_1 and C_1 be the tangent points of ω with the sides BC, CA and AB, respectively. Let A_2, B_2 and C_2 be the tangent points of ω_A with the lines BC, CA and AB. Prove that:

- $\overline{AB} + \overline{BA_2} = \overline{AC} + \overline{CA_2}$;
- $\overline{BA_2} = \overline{CA_1}$, i.e. $\overline{A_1M} = \overline{MA_2}$, where M is the midpoint of BC;



Proof. As tangent segments from the points A, B and C to ω_A, we get

$$\overline{AB_2} = \overline{AC_2}\,, \quad \overline{BA_2} = \overline{BC_2} \quad \text{and} \quad \overline{CA_2} = \overline{CB_2}.$$

$$\therefore \overline{AB} + \overline{BA_2} = \overline{AB} + \overline{BC_2} = \overline{AC_2} = \overline{AB_2} = \overline{AC} + \overline{CB_2} = \overline{AC} + \overline{CA_2} \quad \square$$

Since the sum of both sides equals the whole perimeter of $\triangle ABC$, then each side is equal to its semiperimeter s.

$$\therefore \overline{BA_2} = s - \overline{AB}$$

From Property 10.1.1, we have

$$\overline{AC} + \overline{BA_1} = \overline{AB} + \overline{CA_1}.$$

Again, the sum of both sides equals the whole perimeter of $\triangle ABC$, so each side is equal to its semiperimeter s.

$$\therefore \overline{CA_1} = s - \overline{AB}$$

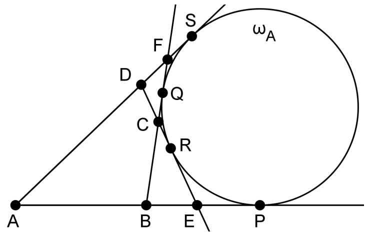
Thus, we can conclude that $\overline{BA_2} = \overline{CA_1}$. Since $\overline{BM} = \overline{CM}$, then we also have $\overline{A_1M} = \overline{MA_2}$. ■

Property 10.1.4 (Ex-tangential quadrilateral)**.** Let $ABCD$ be a quadrilateral such that there exists an excircle ω_A that is tangent to the rays AB (beyond B) and AD (beyond D) and is also tangent to the lines BC and CD. Let E and F be the intersections of the opposite sides. Prove that

$$\overline{AB} + \overline{BC} = \overline{AD} + \overline{DC}$$

$$\overline{EA} + \overline{EC} = \overline{FA} + \overline{FC}$$

$$\overline{EB} + \overline{ED} = \overline{FB} + \overline{FD}.$$



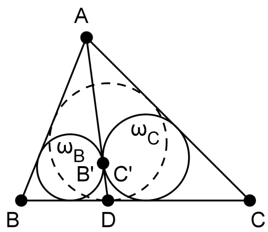
Proof. Let P, Q, R and S be the tangent points of the excircle with the lines AB, BC, CD and DA, respectively. Then, as tangent segments,

$$\overline{AP} = \overline{AS}\,,\ \overline{BP} = \overline{BQ}\,,\ \overline{CQ} = \overline{CR}\,,\ \overline{DR} = \overline{DS}\,,\ \overline{EP} = \overline{ER} \quad \text{and} \quad \overline{FQ} = \overline{FS}.$$

$$\begin{aligned}
\overline{AB} + \overline{BC} &= \overline{AP} - \overline{BP} + \overline{BQ} - \overline{CQ} = \overline{AP} - \overline{CQ} = \\
&= \overline{AS} - \overline{CR} = \overline{AS} - \overline{DS} + \overline{DR} - \overline{CR} = \overline{AD} + \overline{DC} \\
\overline{EA} + \overline{EC} &= \overline{AP} - \overline{EP} + \overline{ER} + \overline{CR} = \overline{AP} + \overline{CR} = \\
&= \overline{AS} + \overline{CQ} = \overline{AS} - \overline{FS} + \overline{FQ} + \overline{CQ} = \overline{FA} + \overline{FC} \\
\overline{EB} + \overline{ED} &= \overline{BP} - \overline{EP} + \overline{ER} + \overline{DR} = \overline{BP} + \overline{DR} = \\
&= \overline{BQ} + \overline{DS} = \overline{BQ} + \overline{FQ} + \overline{DS} - \overline{FS} = \overline{FB} + \overline{FD}
\end{aligned}$$

■

Example 10.1.1. Let ABC be a triangle, and let D be the point where the incircle touches the side BC. Let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Prove that ω_B and ω_C are tangent to each other.



Proof. Let B' and C' be the tangent points of ω_B and ω_C, respectively, to the side AD. We need to prove that $B' \equiv C'$.
Using Property 10.1.1 on $\triangle ABD$ and $\triangle ACD$, we get

$$\overline{AB} + \overline{DB'} = \overline{BD} + \overline{AB'} \quad \text{and} \quad \overline{CD} + \overline{AC'} = \overline{AC} + \overline{DC'}$$

By adding these two equations side by side, we get

$$\overline{AB} + \overline{CD} + \overline{AC'} + \overline{DB'} = \overline{AC} + \overline{BD} + \overline{AB'} + \overline{DC'}$$

From, Property 10.1.1, we know that $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$, so

$$\overline{AC'} + \overline{DB'} = \overline{AB'} + \overline{DC'}.$$

By adding $\overline{AB'} + \overline{AC'}$ on both sides, we get:

$$2 \cdot \overline{AC'} + \overline{AD} = 2 \cdot \overline{AB'} + \overline{AD}$$

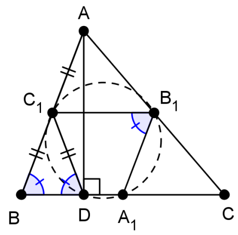
$$\therefore \overline{AC'} = \overline{AB'}, \text{ i.e. } B' \equiv C'$$

■

Related problems: 127, 173 and 190.

10.2 Nine Point Circle

Property 10.2.1. Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB in $\triangle ABC$, respectively. Let D be the foot of the altitude from A to BC. Prove that D lies on the circumcircle of $\triangle A_1B_1C_1$.

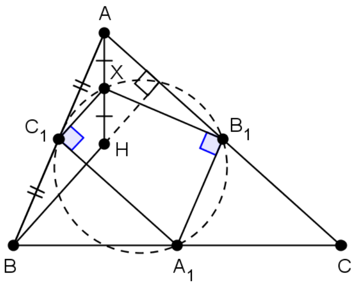


Proof. $\triangle ABD$ is a right triangle and C_1 is the midpoint of the hypotenuse, so $\overline{C_1D} = \overline{C_1B}$. Therefore, $\angle C_1DB = \angle C_1BD = \beta$. B_1C_1 is a midsegment in $\triangle ABC$, so $B_1C_1 \parallel BC$. Similarly, $A_1B_1 \parallel AB$. Therefore, $\angle C_1B_1A_1 = \beta$.

$$\angle C_1B_1A_1 + \angle C_1DA_1 = \beta + (180^\circ - \beta) = 180^\circ$$

$$\therefore D \in (A_1B_1C_1)$$ ■

Property 10.2.2. Let A_1, B_1 and C_1 be the midpoints of the sides BC, CA and AB in $\triangle ABC$, respectively. Let H be the orthocenter in $\triangle ABC$. Let X be the midpoint of AH. Prove that X lies on the circumcircle of $\triangle A_1B_1C_1$.



Proof.

$$C_1X \parallel BH \quad (\because C_1X \text{ is midsegment in } \triangle ABH)$$

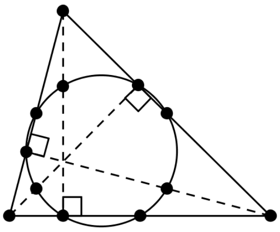
$$BH \perp CA$$

$$CA \parallel C_1A_1 \quad (\because C_1A_1 \text{ is midsegment in } \triangle ABC)$$

$$\therefore C_1X \perp C_1A_1, \text{ i.e. } \angle XC_1A_1 = 90^\circ$$

Similarly, $\angle XB_1A_1 = 90^\circ$. Therefore, $\angle XC_1A_1 + \angle XB_1A_1 = 180^\circ$, so X lies on the circumcircle of $A_1B_1C_1$. ■

With these two properties, we proved that the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments from each vertex to the orthocenter (totally nine points) all lie on one circle. This circle is called the *nine point circle* of the triangle.

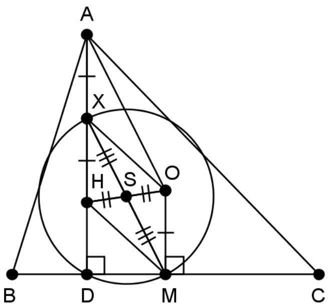


Now, let's try to find the center and the radius of this circle.

Let M be the midpoint of the side BC in $\triangle ABC$. Let H and O be the orthocenter and circumcenter of $\triangle ABC$, respectively. Let X be the midpoint of AH and let D be the foot of the altitude from A. As we know from Property 10.3.2, $\overline{AH} = 2 \cdot \overline{OM}$, so

$$\overline{XH} = \frac{1}{2} \cdot \overline{AH} = \overline{OM}.$$

Also, $XH \parallel OM$ because they are both perpendicular to BC. So $XHMO$ is a parallelogram, which means that the intersection point of its diagonals, let it be S, is their midpoint, i.e. $\overline{HS} = \overline{SO}$ and $\overline{XS} = \overline{SM}$.



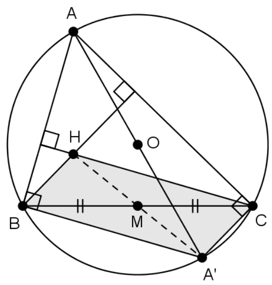
Since D, M and X lie on the nine point circle of $\triangle ABC$ and since $\angle XDM = 90°$, the center of the nine point circle must be on the midpoint of XM, i.e. the point S and SX is a radius in that circle. Also, since SX is midsegment in $\triangle HOA$, $\overline{SX} = \frac{1}{2} \cdot \overline{OA} = \frac{1}{2} \cdot R$. In conclusion,

Property 10.2.3. The center of the nine point circle lies on the Euler line, more specifically it is the midpoint of OH. The radius of the nine point circle is one half of the radius of the circumcircle of $\triangle ABC$.

Related problems: 83, 103 and 114.

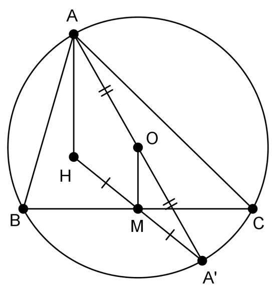
10.3 Euler Line

Property 10.3.1. Let M be the midpoint of the side BC in $\triangle ABC$. Let A' be the antipode of A on the circumcircle, i.e. the point on the circumcircle such that AA' is a diameter. Finally, let H be the orthocenter of $\triangle ABC$. Prove that the points $H - M - A'$ are collinear and $\overline{HM} = \overline{MA'}$.



Proof. Since AA' is a diameter, we have $A'B \perp AB$. But also, as an altitude, $CH \perp AB$, so $A'B \parallel CH$. Similarly, $A'C \parallel BH$. Therefore, $A'BHC$ is a parallelogram, so its diagonals bisect each other. Since M is the midpoint of BC, we get that M is also the midpoint of $A'H$, i.e. $H - M - A'$ are collinear and $\overline{HM} = \overline{MA'}$. ∎

Property 10.3.2. Let M be the midpoint of the side BC in $\triangle ABC$. Let H and O be the orthocenter and circumcenter of $\triangle ABC$, respectively. Prove that $\overline{AH} = 2 \cdot \overline{OM}$.



Proof. Let A' be the antipode of A on the circumcircle. From Property 10.3.1 we know that $H - M - A'$ are collinear and $\overline{HM} = \overline{MA'}$. Also, $\overline{AO} = \overline{OA'}$ as radii. Therefore, OM is midsegment in $\triangle AHA'$, so $\overline{AH} = 2 \cdot \overline{OM}$. ∎

Property 10.3.3 (Euler Line)**.** Let H, T and O be the orthocenter, centroid and circumcenter in $\triangle ABC$, respectively. Prove that the points $H-T-O$ are collinear and $\overline{HT} = 2 \cdot \overline{TO}$.

Proof. Let M be the midpoint of BC. Let $T' = AM \cap HO$. We will prove that $T' \equiv T$. The lines AH and OM are parallel because they are both perpendicular to BC. Therefore, $\triangle AHT' \sim \triangle MOT'$ and

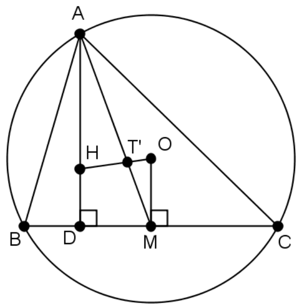
$$\therefore \frac{\overline{AH}}{\overline{MO}} = \frac{\overline{AT'}}{\overline{MT'}} = \frac{\overline{HT'}}{\overline{OT'}}$$

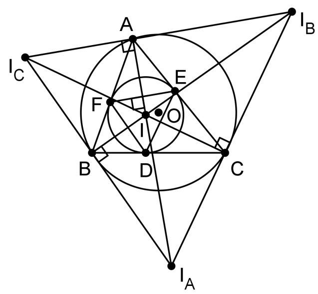
Combining with $\overline{AH} = 2 \cdot \overline{OM}$ (from Property 10.3.2), we get that the ratio of similarity is 2.

$$\therefore \frac{\overline{AT'}}{\overline{T'M}} = 2 : 1,$$

which means that T' is the centroid of the triangle, i.e. $T' \equiv T$. So the points $H-T-O$ are collinear. This line is known as the *Euler line* of $\triangle ABC$.

From the same similarity, we also get that $\overline{HT} = 2 \cdot \overline{TO}$. ■





Property 10.3.4. Let O and I circumcenter and incenter in $\triangle ABC$, respectively. Let D, E and F be tangent points of the incircle with the sides BC, CA and AB, respectively. Prove that IO is the Euler line of $\triangle DEF$.

Proof. Let I_A, I_B and I_C be the $A-$, $B-$ and $C-$excenter of $\triangle ABC$, respectively. From Property 7.4, we know that I is orthocenter of $\triangle I_A I_B I_C$, so $I_B I_C \perp AI$. Since AI is an angle bisector in the isosceles $\triangle AEF$, we have $EF \perp AI$. Therefore, $I_B I_C \parallel EF$ and similarly for the other sides. Therefore, each pair of corresponding sides in triangles $\triangle I_A I_B I_C$ and $\triangle DEF$ are parallel, so they are similar and all of their corresponding elements are also parallel; More precisely, their Euler lines are parallel.

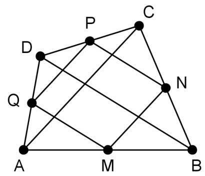
Since A, B and C are the feet of the altitudes in $\triangle I_A I_B I_C$, we get that (ABC) is its nine-point circle and O is its nine-point center. Combining with the fact that I is its orthocenter, we get that IO is the Euler line of $\triangle I_A I_B I_C$.

On the other hand, I is the circumcenter of $\triangle DEF$, so it must lie on its Euler line. Finally, since the Euler line of $\triangle DEF$ must contain I and be parallel to IO, it must be IO. ■

Related problems: 15, 26, 51, 92, 110, 118, 143, 144 and 147.

10.4 Eight Point Circle

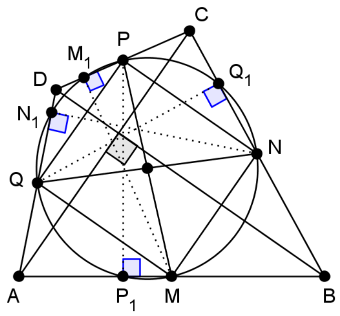
Property 10.4.1. Let $ABCD$ be a convex quadrilateral. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. Prove that $MNPQ$ is a parallelogram.

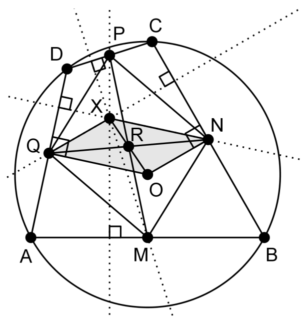


Proof. MN is midsegment in $\triangle ABC$. Therefore, $MN \parallel AC$. Similarly, PQ is midsegment in $\triangle DAC$, so $PQ \parallel AC$. Therefore, $MN \parallel PQ$. Similarly, $MQ \parallel NP$. Therefore, $MNPQ$ is a parallelogram. ■

Property 10.4.2 (Eight Point Circle)**.** Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. Let M_1, N_1, P_1 and Q_1 be the feet of the perpendiculars from M, N, P and Q, respectively, to the opposite sides in the quadrilateral. Prove that the points M, N, P, Q, M_1, N_1, P_1 and Q_1 all lie on a single circle.

Proof. Combining the proof of Property 10.4.1 with $AC \perp BD$, we get that $MNPQ$ is a rectangle, i.e. a quadrilateral inscribed in a circle where the diagonals MP and NQ are diameters. From the definition of M_1, $MM_1 \perp CD$, i.e. $\angle MM_1P = 90°$, so $M_1 \in (MNPQ)$. Similarly, $N_1, P_1, Q_1 \in (MNPQ)$. ■





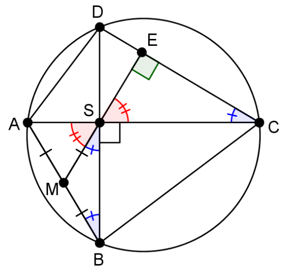
Property 10.4.3. Let $ABCD$ be a cyclic quadrilateral centered at O. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. Let ℓ_M, ℓ_N, ℓ_P and ℓ_Q be the perpendiculars from M, N, P and Q, respectively, to the opposite sides in the quadrilateral. Prove that the lines ℓ_M, ℓ_N, ℓ_P and ℓ_Q are concurrent.

Proof. Let $MP \cap NQ = R$. From Property 10.4.1 we get that $MNPQ$ is a parallelogram, so R is the midpoint of MP and NQ.

Let $\ell_N \cap \ell_Q = X$. Since N is midpoint of the chord BC, we get $ON \perp BC$. Since $QX \perp BC$, we get $ON \parallel QX$. Similarly, $OQ \parallel NX$. Therefore, $ONQX$ is a parallelogram, so its diagonals bisect each other. Since R is midpoint of NQ, we get that R is midpoint of OX, i.e. X is the reflection of O w.r.t. R.

Similarly, if $\ell_M \cap \ell_P = Y$, then Y is the reflection of O w.r.t. to R. Therefore, $X \equiv Y$, i.e. the four perpendiculars are concurrent. ■

Property 10.4.4 (Brahmagupta Theorem)**.** Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals that intersect at S. Let M be the midpoint of the side AB. Prove that $MS \perp CD$.



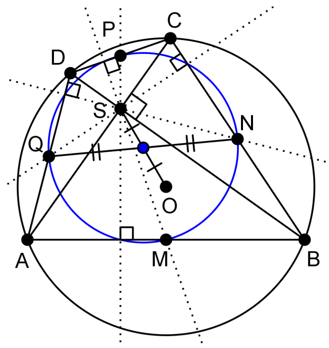
Proof. Let $MS \cap CD = E$. M is the midpoint of the hypotenuse in the right $\triangle ABS$, so $\overline{MB} = \overline{MS}$, i.e. $\angle MBS = \angle MSB$.

$$\angle SCE \equiv \angle ACD = \angle ABD \equiv \angle MBS = \angle MSB$$

$$\angle ESC + \angle SCE = \angle MSA + \angle MSB = \angle ASB = 90°$$

$$\angle SEC = 180° - (\angle ESC + \angle SCE) = 90°, \text{ i.e. } MS \perp CD$$ ■

Property 10.4.5. Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals that intersect at S. Let O be the center of $(ABCD)$. Prove that the eight point circle of $ABCD$ is centered at the midpoint of OS.

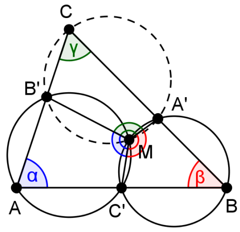


Proof. Let M, N, P and Q be the midpoints of the sides AB, BC, CD and DA, respectively. From Property 10.4.2, we know that NQ is a diameter of the eight point circle, so its center is the midpoint of NQ. From Property 10.4.4, we know that the perpendiculars through the midpoints to the opposite sides pass through the intersection of the diagonals, so S is their concurrency point. In the proof of Property 10.4.3, where this point of concurrency was named X, we already showed that the midpoint of NQ coincides with the midpoint of OS. Therefore, the eight point circle of $ABCD$ is centered at the midpoint of OS. ■

Related problems: 143, 154 and 159.

10.5 Miquel's Theorem

Property 10.5.1 (Miquel's Theorem)**.** Let ABC be a triangle, with arbitrary points A', B' and C' on sides BC, CA and AB, respectively (or their extensions). The circumcircles of $\triangle AB'C'$, $\triangle A'BC'$ and $\triangle A'B'C$ intersect in a single point, called the Miquel point.



Proof. Let's assume that the points A', B' and C' are on the sides (not on the extensions). In the other cases, the proof follows a similar structure.

Let $M = (AB'C') \cap (A'BC')$. We will prove that M lies on $(A'B'C)$, too. Since $AB'MC'$ and $BC'MA'$ are cyclic, we have

$$\angle B'MC' = 180^\circ - \alpha \quad \text{and} \quad \angle C'MA' = 180^\circ - \beta.$$

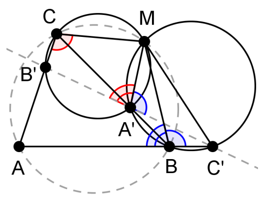
$$\therefore \angle A'MB' = 360^\circ - (\angle B'MC' + \angle C'MA') = \alpha + \beta$$

$$\therefore \angle A'CB' + \angle A'MB' = \gamma + \alpha + \beta = 180^\circ$$

$$\therefore M \in (A'B'C)$$

■

Property 10.5.2. Let ABC be a triangle, with arbitrary points A', B' and C' on sides BC, CA and AB, respectively (or their extensions). The Miquel point lies on the circumcircle of $\triangle ABC$ if and only if the points A', B' and C' are collinear.



Proof. We will see the configuration when one of the points is on the extension of the sides, WLOG let it be C'. The other case, where all three points are on the extensions of the sides follows a similar structure.

Let M be the Miquel point of $\triangle ABC$. Then $MA'BC'$ and $MCB'A'$ are cyclic quadrilaterals.

$$\angle MA'C' = \angle MBC' = 180^\circ - \angle MBA$$

$$\angle MA'B' = 180^\circ - \angle MCB' \equiv 180^\circ - \angle MCA$$

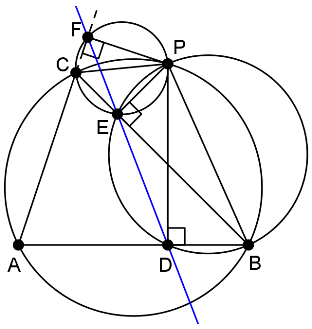
$$\therefore \angle MA'C' + \angle MA'B' = 360^\circ - (\angle MBA + \angle MCA)$$

The points $C' - A' - B'$ are collinear iff the left-hand side is 180°. The right-hand side is 180° iff $ABMC$ is a cyclic quadrilateral, i.e. $M \in (ABC)$. ■

Related problems: 40, 116, 164, 172 and 228.

10.6 Simson Line Theorem

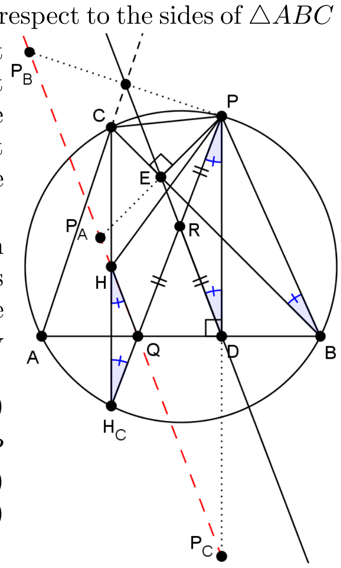
Property 10.6.1 (Simson Line Theorem)**.** Let P be a point on the circumcircle ω of a triangle ABC. If D, E and F are the feet of the perpendiculars from P to the lines AB, BC and CA, prove that the points D, E and F are collinear.



Proof. In Example 8.1, we already gave 3 different proofs of this theorem by angle chasing. Now, we are going to show another proof, from a different perspective.
Since $\angle BDP = 90^\circ = \angle BEP$, we get $BDEP$ is cyclic. Similarly, $CFPE$ is cyclic. By Property 10.5.1, since $P = (DBE) \cap (ECF)$, we get that P is the Miquel Point for $\triangle ABC$ and points D, E, F on its sidelines. Now, by Property 10.5.2, since $P \in (ABC)$, we get that D, E, F must be collinear. ∎

Property 10.6.2. Let P be a point on the circumcircle ω of $\triangle ABC$ and let H be its orthocenter. Prove that the reflections of P with respect to the sides of $\triangle ABC$ are collinear with H.

Proof. From Example 8.1, we know that the feet of perpendiculars from P to the sides of $\triangle ABC$ lie on the P-Simson line of $\triangle ABC$. Then, by Thales' Proportionality Theorem, the reflections of P with respect to the sides of $\triangle ABC$ will also be collinear. We just need to prove that H lies on that line. Since the distance from a point to the foot of the perpendicular to a line is half the distance from the point to its reflection with respect to the line, we need to prove that the P-Simson line bisects the line segment PH.



WLOG let P be on the arc $\widehat{BC}$ that doesn't contain A. Let D and E be the feet of the perpendiculars from P to AB and BC, respectively. Let H_C be the reflection of H with respect to the side AB. By Property 6.8, we know that $H_C \in \omega$.

$DP \parallel H_CH \quad (\because DP \perp AB \wedge H_CH \perp AB) \qquad (1)$

Let $PH_C \cap AB = Q$. Then, $\angle EDP \overset{(PEDB)}{=} \angle EBP \equiv \angle CBP \overset{\omega}{=} \angle CH_CP \equiv \angle HH_CQ = \angle H_CHQ. \quad (2)$

Because of (1), we get that $ED \parallel HQ. \qquad (3)$

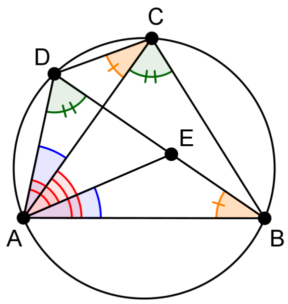
Let $ED \cap PH_C = R$.

Then, $\angle RPD \overset{(1)}{=} \angle RH_CC \equiv \angle PH_CC \overset{(2)}{=} \angle EDP \equiv \angle RDP$. Since $\triangle PDQ$ is right triangle, we can also get $\angle RQD = \angle RDQ$. Therefore, $\overline{RP} = \overline{RD} = \overline{RQ}$, i.e. ED bisects PQ. Combining with (3), we get that the P-Simson line ED bisects the line segment PH. ∎

Related problems: 78, 91 and 130.

10.7 Ptolemy's Theorem

Property 10.7.1 (Ptolemy's Theorem)**.** Let $ABCD$ be a cyclic quadrilateral. Then, $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} = \overline{AC} \cdot \overline{BD}$.



Proof. Let $E \in BD$, such that $\angle CAD = \angle BAE$. Then, also $\angle EAD = \angle BAC$. As inscribed angles over the same arc in ω, we have $\angle ACD = \angle ABD \equiv \angle ABE$ and $\angle ADE \equiv \angle ADB = \angle ACB$. By AA we get $\triangle CAD \sim \triangle BAE$ and $\triangle EAD \sim \triangle BAC$. Thus,

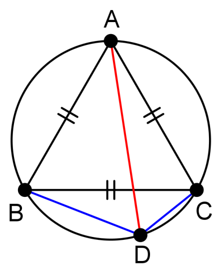
$$\frac{\overline{CA}}{\overline{BA}} = \frac{\overline{CD}}{\overline{BE}} \iff \overline{AB} \cdot \overline{CD} = \overline{AC} \cdot \overline{BE}$$

$$\frac{\overline{ED}}{\overline{BC}} = \frac{\overline{AD}}{\overline{AC}} \iff \overline{BC} \cdot \overline{AD} = \overline{AC} \cdot \overline{ED}.$$

Therefore,

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} = \overline{AC} \cdot (\overline{BE} + \overline{ED}) = \overline{AC} \cdot \overline{BD} \qquad \blacksquare$$

Example 10.7.1. Let ABC be an equilateral triangle with circumcircle ω. If $D \in \widehat{BC}$ on ω, prove that $\overline{DA} = \overline{DB} + \overline{DC}$.



Proof. Applying Ptolemy's Theorem for the cyclic quadrilateral $ABDC$, we get:

$$\overline{AB} \cdot \overline{DC} + \overline{BD} \cdot \overline{AC} = \overline{AD} \cdot \overline{BC},$$

but since $\overline{AB} = \overline{BC} = \overline{AC}$, we get

$$\overline{DC} + \overline{BD} = \overline{AD}. \qquad \blacksquare$$

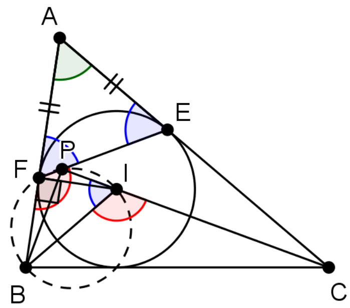
Related problem: 141.

10.8 In-Touch Chord

Property 10.8.1. Let I be the incenter of $\triangle ABC$ and let E and F be the tangent points of the incircle with the sides AC and AB, respectively. Let $CI \cap EF = P$. Then, $BP \perp PC$.

Proof. Because I is the incenter and F is the tangent point of AB and the incircle, we have $\angle BFI = 90°$. We want to prove that $\angle BPC \equiv \angle BPI = 90°$, so we need to prove that the quadrilateral $BFPI$ is cyclic. From $\triangle BIC$, we get $\angle BIC = 180° - (\frac{\alpha+\beta}{2}) = 90° + \frac{\alpha}{2}$.
Therefore,

$$\angle BIP = 90° - \frac{\alpha}{2}.$$



Because AE and AF are tangent to the incircle, as tangent segments, they are of equal length and therefore $\triangle AEF$ is isosceles.

$$\therefore \angle AFE = \angle AEF = 90° - \frac{\alpha}{2}$$

$$\therefore \angle BFP = 180° - \angle AFE = 90° + \frac{\alpha}{2}.$$

Finally, $\angle BIP + \angle BFP = 180°$ and thus $BFPI$ is cyclic.

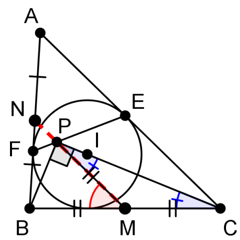
$$\therefore \angle BPC \equiv \angle BPI = \angle BFI = 90°$$ ■

Property 10.8.2. The A-intouch chord, B-midsegment and $\angle C$-bisector are concurrent.

Proof. Let I be the incenter of $\triangle ABC$ and let E and F be the tangent points of the incircle with the sides AC and AB, respectively. Let $CI \cap EF = P$. Let M and N be the midpoints of BC and BA, respectively. We will prove that $P \in MN$.

From Property 10.8.1 we know that $\triangle BPC$ is right-angled. Since M is the midpoint of its hypotenuse, we get that $\overline{MB} = \overline{MP} = \overline{MC}$. Therefore, as an exterior angle in $\triangle MCP$

$$\angle BMP = 2\angle MCP \equiv 2\angle BCI = 2 \cdot \frac{\gamma}{2} = \gamma$$



Also, MN is midsegment in $\triangle ABC$, so $MN \parallel AC$.

$$\therefore \angle BMN = \angle BCA = \gamma.$$

Finally, $\angle BMP = \angle BMN$, so $M - P - N$ are collinear, i.e. $P \in MN$. ■

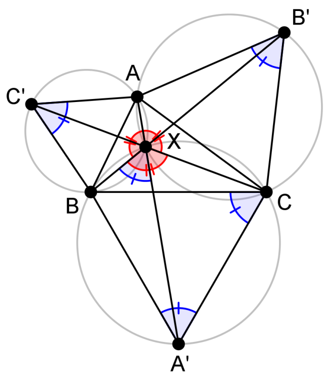
Related problems: 106, 108, 120, 129, 146 and 217.

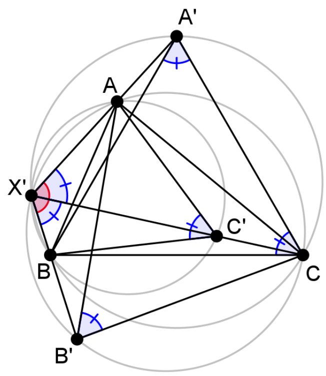
10.9 Fermat Points

Property 10.9.1 (First Fermat Point)**.** Let ABC be a triangle. Construct equilateral triangles $\triangle ABC'$, $\triangle BCA'$ and $\triangle CAB'$ outward $\triangle ABC$. Then, AA', BB' and CC' are concurrent.

Proof. Let $(ABC') \cap (CAB') = X$. Then, $\angle AXB = 180° - \angle AC'B = 120°$. Similarly, $\angle CXA = 120°$. Therefore,
$\angle BXC = 360° - (\angle CXA + \angle AXB) = 120° = 180° - \angle BA'C$, so $X \in (BCA')$.

We will prove that AA', BB' and CC' pass through X. We have $\angle AXB + \angle BXA' = 120° + \angle BCA' = 180°$, so $X \in AA'$. Similarly, $X \in BB'$ and $X \in CC'$. The point X is known as the *First Fermat Point* of $\triangle ABC$. ∎





Property 10.9.2 (Second Fermat Point)**.** Let ABC be a triangle. Construct equilateral triangles $\triangle ABC'$, $\triangle BCA'$ and $\triangle CAB'$ inward $\triangle ABC$. Then, AA', BB' and CC' are concurrent.

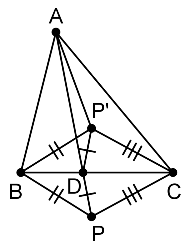
Proof. Let $(ABC') \cap (CAB') = X'$. Then, $\angle AX'B = 180° - \angle AC'B = 120°$ and $\angle AX'C = \angle AB'C = 60°$. Therefore,
$\angle BX'C = \angle BX'A - \angle AX'C = 120° - 60° = 60° = \angle BA'C$, so $X' \in (BCA')$.

We will prove that AA', BB' and CC' pass through X'. We have $\angle BX'A = 120° = 180° - \angle BCA' = \angle BX'A'$, so $X' - A - A'$ are collinear. Similarly, $X' \in BB'$ and $X' \in CC'$. The point X' is known as the *Second Fermat Point* of $\triangle ABC$. ∎

Property 10.9.3 (Fermat's Problem / Torricelli Point)**.** Find the point P in the plane of $\triangle ABC$ that minimizes $\overline{PA} + \overline{PB} + \overline{PC}$.

Proof. Let P be a point outside $\triangle ABC$. Then, for at least one sideline, P and the corresponding vertex must be on different sides of that sideline. WLOG, let P and A be on different sides of the line BC. Let P' be the reflection of P w.r.t. BC and let $AP \cap BC = D$. Then, using Property 2.14 and Triangle Inequality in $\triangle ADP'$, we get:

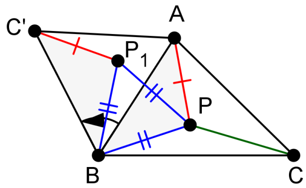
$$\overline{PA} + \overline{PB} + \overline{PC} = \overline{PD} + \overline{DA} + \overline{PB} + \overline{PC} =$$
$$= \overline{P'D} + \overline{DA} + \overline{P'B} + \overline{P'C} > \overline{P'A} + \overline{P'B} + \overline{P'C},$$

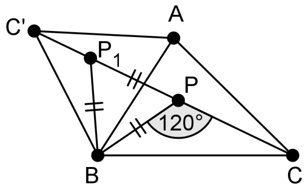


so P' is a point with smaller sum of distances than P. Therefore, the desired point P can not be outside $\triangle ABC$.

Now, let P be a point inside $\triangle ABC$ and let the rotation centered at B with angle 60° send $\triangle BPA$ to $\triangle BP_1C'$. Then, $\triangle BPA \cong \triangle BP_1C'$ and $\angle PBP_1 = 60^\circ = \angle ABC'$, so $\triangle PBP_1$ is equilateral. So,

$$\overline{PA} + \overline{PB} + \overline{PC} = \overline{C'P_1} + \overline{P_1P} + \overline{PC} \geq \overline{C'C}.$$



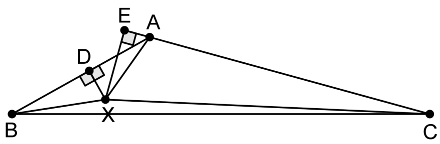


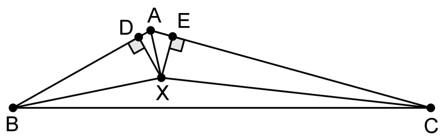
The minimum is achieved when $P \in CC'$. But since $\triangle ABC'$ is the equilateral triangle erected outward on AB, and since similarly we can get that the minimum is achieved when $P \in AA', BB'$, we get that the desired point P is the First Fermat Point. Also, from the fact that the minimum is achieved when $P, P_1 \in CC'$, we get that $\angle BPC = 180^\circ - \angle BPP_1 = 120^\circ$. If $\angle BAC > 120^\circ$, then $\angle BAC > \angle BPC$, so by Property 5.9 we get that P is outside (ABC) and therefore outside $\triangle ABC$, which contradicts our assumption. Therefore, when all angles of $\triangle ABC$ are less than or equal to 120°, the desired point P is the First Fermat Point.

Finally, we will prove that if one angle in $\triangle ABC$ is greater than 120°, then the desired point P is the vertex of that angle. WLOG let $\angle BAC > 120^\circ$. Let X be any point inside $\triangle ABC$. We need to prove that $\overline{XA}+\overline{XB}+\overline{XC} > \overline{AB}+\overline{AC}$. Let D, E be the feet of the perpendiculars from X to AB, AC, respectively. We have two cases:

i) One of the feet, WLOG E, lies on the extension of the side.

Since the hypotenuse is the longest side in a right triangle, we get $\overline{XA} > \overline{AD}$, $\overline{XB} > \overline{DB}$ and $\overline{XC} > \overline{EC} > \overline{AC}$. □

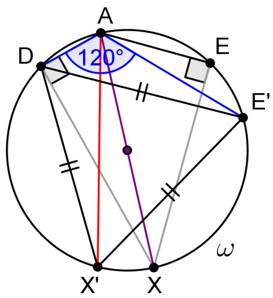




ii) Both D, E lie on the line segments AB, AC.

We have $\overline{AB} = \overline{AD} + \overline{DB}$, $\overline{AC} = \overline{AE} + \overline{EC}$, $\overline{XB} > \overline{DB}$ and $\overline{XC} > \overline{EC}$. We are left to prove that $\overline{XA} > \overline{AD} + \overline{AE}$.

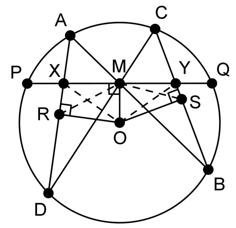
Since $\angle ADX = 90^\circ = \angle AEX$, we get $ADEX$ is cyclic with diameter AX and let ω be its circumcircle. Let $E' \in \omega$, such that $\angle DAE' = 120^\circ$. Since $\angle DAE > \angle DAE'$, E lies on the shorter arc $\widehat{AE'}$ and thus $\overline{AE'} > \overline{AE}$. Let $X' \in \omega$, such that $\triangle DE'X'$ is equilateral. Since the diameter is the longest chord in a circle, we get $\overline{XA} \geq \overline{X'A}$. From Example 10.7.1, we know that $\overline{AX'} = \overline{AD} + \overline{AE'}$. Therefore, $\overline{XA} \geq \overline{X'A} = \overline{AD} + \overline{AE'} > \overline{AD} + \overline{AE}$. ■



10.10 Butterfly Theorem

Property 10.10.1 (Butterfly Theorem)**.** Let M be the midpoint of a chord PQ of a circle ω, through which two other chords AB and CD are drawn. Let $AD \cap PQ = X$ and $BC \cap PQ = Y$. Prove that M is also the midpoint of XY.

Proof. Let O be the center of ω. Since M is the midpoint of PQ, $OM \perp PQ$. Thus, in order to show that $\overline{XM} = \overline{MY}$, we need to prove that $\angle MOX = \angle MOY$.



Let R and S be the feet of the perpendiculars from O to AD and BC, respectively. Therefore, $\overline{AR} = \overline{RD}$ and $\overline{BS} = \overline{SC}$.

$$\angle DAM \equiv \angle DAB \overset{\omega}{=} \angle DCB \equiv \angle MCB \quad \text{and} \quad \angle AMD = \angle CMB,$$

$$\therefore \triangle AMD \sim \triangle CMB$$

$$\therefore \frac{\overline{AD}}{\overline{AM}} = \frac{\overline{CB}}{\overline{CM}}$$

$$\therefore \frac{\overline{AR}}{\overline{AM}} = \frac{\overline{CS}}{\overline{CM}} \quad (\because \frac{\overline{AD}}{\overline{AR}} = 2 = \frac{\overline{CB}}{\overline{CS}})$$

$$\therefore \triangle AMR \sim \triangle CMS \quad (\because \angle RAM \equiv \angle DAB \overset{\omega}{=} \angle DCB \equiv \angle MCS)$$

$$\therefore \angle MRA = \angle MSC \tag{*}$$

Since $OM \perp PQ$ and $OR \perp AD$, $\angle ORX + \angle OMX \equiv \angle ORA + \angle OMP = 180°$. Therefore, $OMXR$ is cyclic. Similarly, $OMYS$ is cyclic. Therefore,

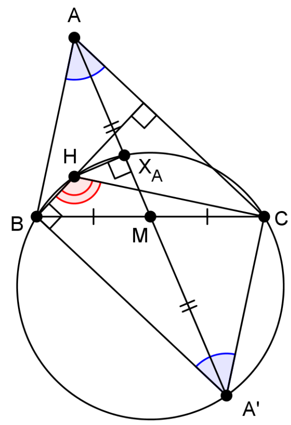
$$\angle MOX = \angle MRX \equiv \angle MRA \overset{(*)}{=} \angle MSC \equiv \angle MSY = \angle MOY \qquad \blacksquare$$

Related problems: 89, 95 and 105.

10.11 HM Point

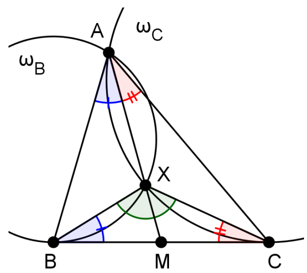
This section is about a set of points that have many properties, but still have no official name; On the Internet, they are known as the "HM points" (there are 3 in every triangle). In a triangle ABC, the $A-HM$ point, denoted by X_A, is the foot of the perpendicular from the orthocenter H to the median AM.

Property 10.11.1. Let ABC be a triangle with orthocenter H. Prove that the point X_A lies on the circumcircle of $\triangle BHC$.



Proof. Let M be the midpoint of the side BC and let $A' \in AM$, such that $\overline{AM} = \overline{MA'}$. Then $ABA'C$ is a parallelogram.
Since the opposite angles in a parallelogram are equal, we have $\angle BA'C = \angle BAC = \alpha$. We know that $\angle BHC = 180° - \alpha$. Therefore, $\angle BA'C + \angle BHC = 180°$, i.e. $A' \in (BHC)$.
Since $BH \perp AC$ and $AC \parallel BA'$, we get $BH \perp BA'$ and therefore $\angle HBA' + \angle HX_AA' = 180°$, i.e. $X_A \in (HBA'C)$. ∎

Property 10.11.2. Let ABC be a triangle and let ω_B be the circle that passes through A and B and is tangent to the line BC. Similarly, let ω_C be the circle that passes through A and C and is tangent to the line BC. Prove that the second intersection of ω_B and ω_C is X_A.



Proof. Let X be the second intersection of ω_B and ω_C. We will prove that $X \equiv X_A$ by proving that X lies on the $A-$median and on the circumcircle of $\triangle BHC$, where H is the orthocenter of $\triangle ABC$.

Let $AX \cap BC = M$. From Secant-Tangent Theorem we get that $\overline{MB}^2 = \overline{MX} \cdot \overline{MA} = \overline{MC}^2$, so M is midpoint of BC, i.e. X lies on the $A-$median.

Since BM is tangent to ABX, we get that $\angle MBX = \angle BAX$. Similarly, $\angle MCX = \angle CAX$. Therefore, from $\triangle BXC$,

$$\angle BXC = 180° - (\angle XBC + \angle XCB) = 180° - (\angle BAX + \angle CAX) = 180° - \alpha.$$

We also know that $\angle BHC = 180° - \alpha$, so $X \in (BHC)$. ∎

You can find many more properties of the HM point, solved example problems and unsolved exercises in [1]. Some of these are more advanced, so you may want to finish the remaining chapters in this book before trying them.

Related problems: 138 and 167.

Chapter 11

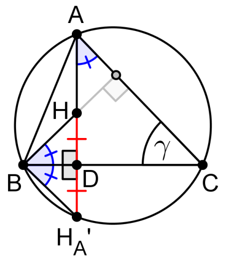
Phantom Points

When we struggle to prove something for a point X directly with the conditions we are given, sometimes it's useful to define a *phantom point* X' that already satisfies the thing we need to prove and then prove that X' also satisfies the given conditions, so it must be the same point as X.

We will start by giving an alternate proof of Property 6.8.

Example 11.1. Let H_A be the reflection of the orthocenter H with respect to the side BC in $\triangle ABC$. Prove that H_A lies on (ABC).

Proof. A point that already satisfies the thing we need to prove would lie on (ABC). So, let ${H_A}' = AH \cap (ABC)$. Now, in order to prove that ${H_A}' \equiv H_A$, we need to show that ${H_A}'$ is the reflection of H w.r.t. BC. Since $H{H_A}' \perp BC$, we only need to show that $\overline{D{H_A}'} = \overline{DH}$, where $D = AH \cap BC$.

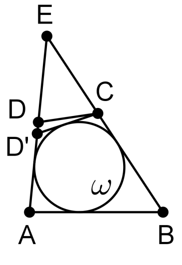


$$\angle {H_A}'BD \equiv \angle {H_A}'BC = \angle {H_A}'AC \equiv \angle HAC =$$
$$= 90° - \gamma = \angle HBC \equiv \angle HBD$$

Combining this with $\angle BD{H_A}' = 90° = \angle BDH$ and the fact that BD is a common side, by ASA we get $\triangle BD{H_A}' \cong \triangle BDH$. Therefore, $\overline{D{H_A}'} = \overline{DH}$. ■

Example 11.2. Let $ABCD$ be a convex quadrilateral such that $\overline{AB} + \overline{CD} = \overline{BC} + \overline{DA}$. Prove that $ABCD$ is a tangential quadrilateral, i.e. a circle can be inscribed inside it.

Proof. If $ABCD$ is a parallelogram, then from the given condition we get that it must be a rhombus and the rhombus is a tangential quadrilateral.



Otherwise, at least one pair of the opposite sidelines intersect. WLOG, $BC \cap AD = E$ and WLOG E lies on the same side of AB as C and D. Let ω be the incircle of $\triangle ABE$. It is already tangent to AB, $BE \equiv BC$ and $EA \equiv DA$. We need to prove that CD is tangent to ω. Let the tangent from C to ω intersect line AD at point D'. We will show that $D' \equiv D$.

From the given condition and using Property 10.1.2 for the tangential quadrilateral $ABCD'$, we get $\overline{AB}+\overline{CD}=\overline{BC}+\overline{DA}$ and $\overline{AB}+\overline{CD'}=\overline{BC}+\overline{D'A}$. WLOG, let D' be between A and D. We subtract the second equation from the first one and we get $\overline{CD}-\overline{CD'}=\overline{DA}-\overline{D'A}=\overline{DD'}$, i.e. $\overline{CD}=\overline{DD'}+\overline{CD'}$, so by Triangle Inequality in $\triangle CDD'$, $D'\in CD$. Since by definition $D'\in AD$, we get $D'=CD\cap AD=D$. ■

Example 11.3. Let ABC be a right triangle and let M,N be midpoints of the legs CA,CB, respectively. The circumcircle of $\triangle CMN$ intersects AB at points P,Q, such that $A-P-Q-B$. If $\overline{CA}=6$ and $\overline{CB}=8$, find $\overline{PQ}$.

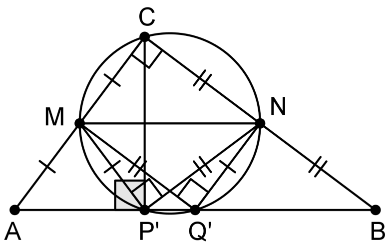
Proof. If we draw an accurate diagram, we can notice "which" points are P and Q, but it may be difficult to prove it directly. So, let P' be the foot of the $C-$altitude and let Q' be the midpoint of AB. We will show that $P\equiv P'$ and $Q\equiv Q'$ by showing that they lie on $(CMN)\equiv\omega$.

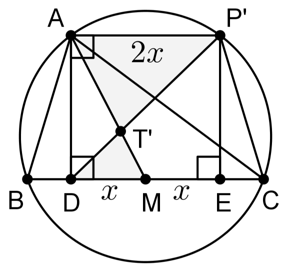
Since $\triangle CAP'$ is a right triangle and M is the midpoint of its hypotenuse, by Property 2.21 we get that $\overline{MP'}=\overline{MA}=\overline{MC}$. Similarly, $\overline{NP'}=\overline{NB}=\overline{NC}$.

Since M,N,Q' are midpoints of CA,CB,AB, we get that $Q'M$ and $Q'N$ are midsegments in $\triangle ABC$ and therefore, by Property 2.20 $\overline{Q'M}=\overline{NC}$ and $\overline{Q'N}=\overline{MC}$.

Therefore, by the criterion SSS, $\triangle CMN\cong\triangle P'MN\cong\triangle Q'NM$ and thus $\angle MP'N=\angle NQ'M=\angle MCN=90^\circ$, so by Thales' Theorem $P',Q'\in\omega$.

Now, there are many ways to finish. Here is one: from Pythagorean Theorem in $\triangle ABC$, we get $\overline{AB}=\sqrt{6^2+8^2}=10$. From Euclid's laws, we have $\overline{AC}^2=\overline{AP}\cdot\overline{AB}$, so $\overline{AP}=3.6$. Finally, $\overline{PQ}=\overline{AQ}-\overline{AP}=\frac{\overline{AB}}{2}-\overline{AP}=1.4$. ■





Example 11.4 (Lemma from Serbia JMO 2017)**.** Let T be the centroid of $\triangle ABC$ and let D be the foot of the $A-$altitude. The ray DT intersects the circumcircle of $\triangle ABC$ at P. Prove that $AP\parallel BC$.

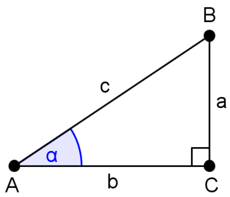
Proof. Let $P'\in(ABC)$, such that $AP'\parallel BC$. Then, by Property 5.7, $\widehat{AB}=\widehat{P'C}$ and thus $\overline{AB}=\overline{P'C}$. So, $ABCP'$ is an isosceles trapezoid. Let E be the foot of the perpendicular from P' to BC. Then, if M is the midpoint of BC, we have $\overline{MD}=\overline{ME}=x$. Since $ADEP'$ is rectangle, we have $\overline{AP'}=\overline{DE}=2x$. Let $AM\cap DP'=T'$. Since $AP'\parallel DM$, by AA we get $\triangle T'AP'\sim\triangle T'MD$. Therefore, $\frac{\overline{T'A}}{\overline{T'M}}=\frac{\overline{AP'}}{\overline{MD}}=\frac{2x}{x}=2$. Thus, T' divides the median AM in ratio $2:1$, so it must be the centroid, i.e. $T'\equiv T$. Now, $D-T-P'$ are collinear and $P'\in(ABC)$, so by definition of P, $P'\equiv P$. ■

Related problem: 147.

Chapter 12

Basic Trigonometry

Trigonometric Functions in Right Triangle



Let ABC be a right triangle ($\gamma = 90°$). Then, we define the sine and cosine functions as follows:

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

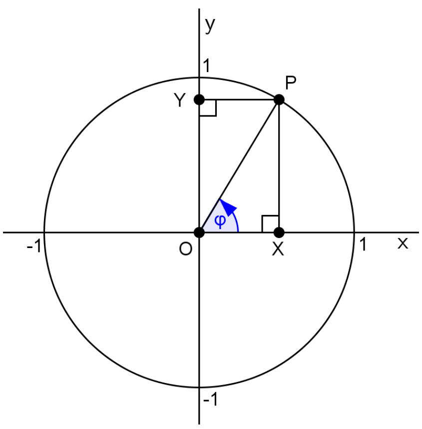
$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

Although they may sound scary, they are nothing more than ratios of sides :)

The Unit Circle

We defined the basic trigonometric functions for angles $0° < \varphi < 90°$. Let's try to extend them for all values of φ.

Let's take a look at the unit circle. That is a circle which is centered at the origin $O\,(0,\,0)$ of the coordinate plane and has a radius of length 1. Let's represent any angle φ with a point P on the unit circle, such that the angle starting from the positive x-axis and going in the counter-clockwise direction to the line OP is equal to φ.



Let $0° < \varphi < 90°$. Let P be a point on the unit circle that represents the angle φ. Let X and Y be the feet of the perpendicular from P to the x- and y-axis, respectively. Then the triangle $\triangle OPX$ is a right triangle with hypotenuse $\overline{OP} = 1$, so by the definitions above, we get that $\cos\varphi = \overline{OX}$ and $\sin\varphi = \overline{PX} = \overline{OY}$. That's right, the cosine and sine values are in fact the x- and y-component of the point P in the coordinate system.

So why not extend this defintion for all possible values of φ? Those are, in fact, the actual definitions for the cosine and sine functions. The cosine is the x-component and the sine is the y-component of the point P representing the angle φ. For example, $\cos(120°) = -\frac{1}{2}$ and $\sin(90°) = 1$. As we can see, the ranges of both the cosine and sine functions are $[-1, 1]$.

Using the unit circle and the definition above, very simply, using congruence of triangles, or the Pythagorean Theorem, we can prove various properties, like:

$$\sin(180° - \alpha) = \sin\alpha \qquad \cos(180° - \alpha) = -\cos\alpha$$

$$\sin(90° + \alpha) = \cos\alpha \qquad \cos(90° + \alpha) = -\sin\alpha$$

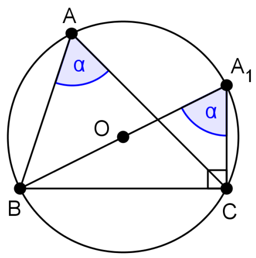
$$\cos^2\alpha + \sin^2\alpha = 1$$

Property 12.1 (Law of Sines)**.** In a triangle $\triangle ABC$ with circumradius R,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$

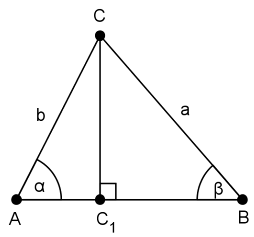
Proof. Let ω be the circumcirle of $\triangle ABC$ and let O be its center. Let A_1 be the second intersection of BO and ω. Then $\angle BA_1C = \angle BAC = \alpha$. On the other hand, $\angle A_1CB = 90°$ as an inscribed angle over the diameter, so $\triangle A_1BC$ is a right triangle. By definition,

$$\sin\alpha = \frac{\overline{BC}}{\overline{A_1B}} = \frac{a}{2R}, \text{ i.e. } \frac{a}{\sin\alpha} = 2R \qquad \blacksquare$$



Property 12.2 (Law of Cosines)**.** In a triangle $\triangle ABC$, for any side

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$



Proof. Let C_1 be the feet of the altitude from C to AB. Let's investigate the case when C_1 is between A and B. From the two right triangles $\triangle ACC_1$ and $\triangle BCC_1$ we get $\overline{AC_1} = b\cos\alpha$ and $\overline{BC_1} = a\cos\beta$. Since $\overline{AB} = \overline{AC_1} + \overline{BC_1}$, we get

$$c = a\cos\beta + b\cos\alpha.$$

We get exactly the same result even when C_1 is not between A and B because of the property $\cos(180° - \alpha) = -\cos\alpha$. Multiplying the last equation by c on both sides, we get

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

Similarly, we can get

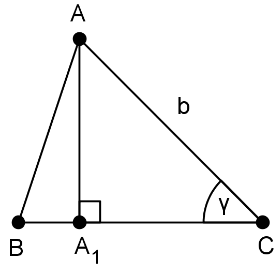
$$ab\cos\gamma + ac\cos\beta = a^2$$

$$ab\cos\gamma + bc\cos\alpha = b^2$$

By adding the last three equations side by side, we get the desired result. $\blacksquare$

Property 12.3. The area of a triangle $\triangle ABC$ can be expressed as

$$P_{\triangle ABC} = \frac{1}{2}ab\sin\gamma.$$



Proof. Let A_1 be the feet of the altitude from A to BC. Then $\triangle CAA_1$ is a right triangle, so we have

$$\sin\gamma = \frac{\overline{AA_1}}{\overline{AC}}, \text{ i.e. } \overline{AA_1} = b\sin\gamma.$$

Then, the area of $\triangle ABC$ is

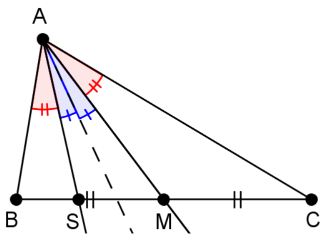
$$P_{\triangle ABC} = \frac{1}{2}\cdot\overline{BC}\cdot\overline{AA_1} = \frac{1}{2}ab\sin\gamma. \qquad \blacksquare$$

Related problems: 27 and 204.

Chapter 13

Symmedian

Symmedian is the reflection of the median across the corresponding angle bisector.



We will now see and prove a few properties of the symmedians.

Property 13.1. The symmedian AS divides the opposite side in the ratio of the square of the sides, i.e.

$$\frac{\overline{BS}}{\overline{CS}} = \left(\frac{\overline{AB}}{\overline{AC}}\right)^2.$$

Proof. We can express the area of $\triangle BAS$ in two ways:

$$\frac{1}{2} \cdot \overline{BS} \cdot h_a = P_{\triangle BAS} = \frac{1}{2} \cdot \overline{BA} \cdot \overline{AS} \cdot \sin(\angle BAS)$$

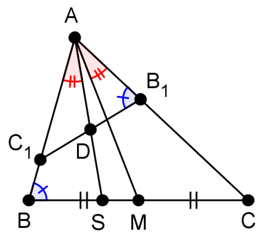
Since the symmedian is the reflection of the median with respect to the angle bisector, we have $\angle BAS = \angle CAM$ and $\angle BAM = \angle CAS$. By expressing the areas of the other triangles in two ways, similarly, we get:

$$\frac{\overline{BS}}{\overline{MC}} = \frac{P_{\triangle BAS}}{P_{\triangle MAC}} = \frac{\overline{BA} \cdot \overline{AS}}{\overline{AM} \cdot \overline{AC}}$$

$$\frac{\overline{BM}}{\overline{SC}} = \frac{P_{\triangle BMA}}{P_{\triangle CSA}} = \frac{\overline{BA} \cdot \overline{AM}}{\overline{AS} \cdot \overline{AC}}$$

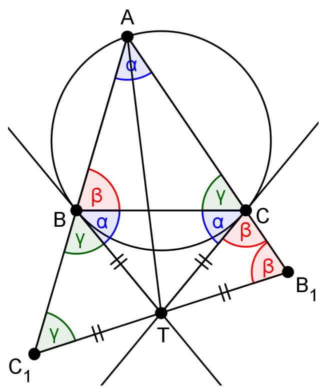
By multiplying these equalities, we are done. ∎

Property 13.2. A symmedian drawn from a vertex of a triangle bisects the antiparallels to the opposite side (with respect to the adjacent sides).



Proof. Let AS and AM be the symmedian and the median from the vertex A in $\triangle ABC$, respectively. Let $B_1 \in AC$ and $C_1 \in AB$, such that B_1C_1 is antiparallel to BC with respect to the lines AB and AC, i.e. $\angle AB_1C_1 = \angle ABC$. Therefore, $\triangle ABC \sim \triangle AB_1C_1$. Let $AS \cap B_1C_1 = D$. By the definition of symmedian, $\angle BAS = \angle CAM$, which means that the similarity "maps" AM in $\triangle ABC$ to $AS \equiv AD$ in $\triangle AB_1C_1$. Therefore, AD is median in $\triangle AB_1C_1$, i.e. the symmedian AS bisects B_1C_1 which is antiparallel to the opposite side BC. ∎

Property 13.3. Given a triangle ABC and its circumcircle, let the intersection of the tangents at the points B and C intersect at T. Then, AT is a symmedian in $\triangle ABC$.



Proof. Since the angle between a tangent and a chord is equal to any inscribed angle over the same chord, $\angle CBT = \angle CAB = \alpha$ and $\angle BCT = \angle BAC = \alpha$, so $\triangle BCT$ is isosceles and therefore $\overline{TB} = \overline{TC}$. Let $B_1 \in AC$ and $C_1 \in AB$, such that B_1C_1 is an antiparallel line to BC (with respect to the lines AB and AC) that passes through T. Then, $\angle AB_1C_1 = \angle ABC = \beta$.

Now, $\angle TCB_1 = 180^\circ - (\angle ACB + \angle BCT) = 180 - (\gamma + \alpha) = \beta$ and $\angle TB_1C \equiv \angle C_1B_1A = \beta$, so $\triangle TCB_1$ is isosceles, i.e. $\overline{TC} = \overline{TB_1}$. Similarly, $\overline{TB} = \overline{TC_1}$.

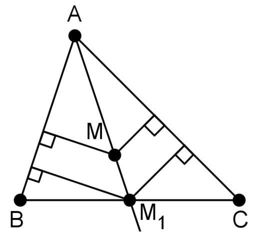
In conclusion, $\overline{TC_1} = \overline{TB} = \overline{TC} = \overline{TB_1}$, so T is the midpoint of B_1C_1. By Property 13.2, it follows that AT is the symmedian from the vertex A in $\triangle ABC$. ∎

Property 13.4. The A-symmedian is the locus of the points P such that

$$\frac{d(P,\, AB)}{d(P,\, AC)} = \frac{\overline{AB}}{\overline{AC}}.$$

Proof. We will firstly prove that the median is the locus of the points M such that

$$\frac{d(M,\, AB)}{d(M,\, AC)} = \frac{\overline{AC}}{\overline{AB}}.$$



Let M be a point in the interior of $\angle BAC$. Let AM meet BC at M_1. By similarity of triangles, we get that

$$\frac{d(M,\, AB)}{d(M_1,\, AB)} = \frac{\overline{AM}}{\overline{AM_1}} = \frac{d(M,\, AC)}{d(M_1,\, AC)}.$$

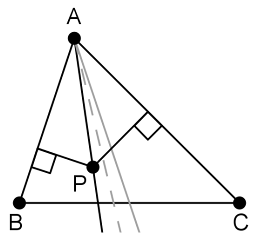
By rearranging, we get

$$\frac{d(M_1,\, AB)}{d(M_1,\, AC)} = \frac{d(M,\, AB)}{d(M,\, AC)} = \frac{\overline{AC}}{\overline{AB}}$$

$$\iff d(M_1,\, AB) \cdot \overline{AB} = d(M_1,\, AC) \cdot \overline{AC}$$

$$\iff P_{\triangle ABM_1} = P_{\triangle ACM_1}$$

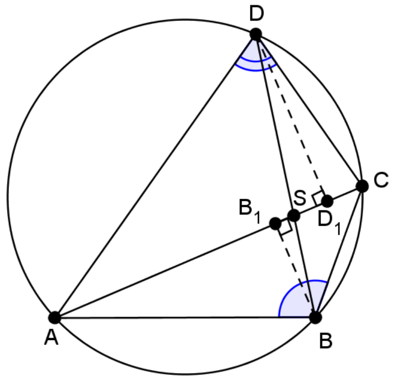
$$\iff \overline{BM_1} = \overline{M_1C} \quad \square$$



Since the symmedian is the reflection of the median with respect to the angle bisector, by symmetry we have that it is the locus of the points P such that

$$\frac{d(P,\, AB)}{d(P,\, AC)} = \frac{d(M,\, AC)}{d(M,\, AB)} = \frac{\overline{AB}}{\overline{AC}} \qquad \blacksquare$$

Property 13.5. Given a cyclic quadrilateral $ABCD$, let S be the intersection of the diagonals AC and BD. Then, the diagonal AS is a symmedian in $\triangle ABD$ if and only if $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$.



Proof. Firstly, let's investigate something that is true in any cyclic quadrilateral. Let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to AC. Then, $\triangle BB_1S \sim \triangle DD_1S$. Also, $\sin\beta = \sin(180° - \delta) = \sin\delta$.

$$\frac{\overline{BA} \cdot \overline{BC}}{\overline{DA} \cdot \overline{DC}} = \frac{\frac{1}{2}\,\overline{BA}\,\overline{BC}\sin\beta}{\frac{1}{2}\,\overline{DA}\,\overline{DC}\sin\delta} = \frac{P_{\triangle ABC}}{P_{\triangle ADC}} = \frac{\frac{1}{2}\,\overline{AC}\,\overline{BB_1}}{\frac{1}{2}\,\overline{AC}\,\overline{DD_1}} = \frac{\overline{BB_1}}{\overline{DD_1}} = \frac{\overline{BS}}{\overline{SD}} \qquad (*)$$

Then, by Property 13.1,

$$\frac{\overline{BS}}{\overline{SD}} = \left(\frac{\overline{AB}}{\overline{AD}}\right)^2 \overset{(*)}{\iff} \frac{\overline{BA} \cdot \overline{BC}}{\overline{DA} \cdot \overline{DC}} = \left(\frac{\overline{AB}}{\overline{AD}}\right)^2 \iff \frac{\overline{BC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{AD}} \qquad \blacksquare$$

Remark. Since the previous property works in both directions, we can conclude that AS is a symmedian in $\triangle ABD$ if and only if BS is a symmedian in $\triangle ABC$, i.e. if a diagonal in a cyclic quadrilateral is a symmedian in one of the four triangles, then a diagonal is a symmedian in each of the four triangles.

Now, we will present a problem with many different solutions to see how we can use these properties in an Olympiad problem.

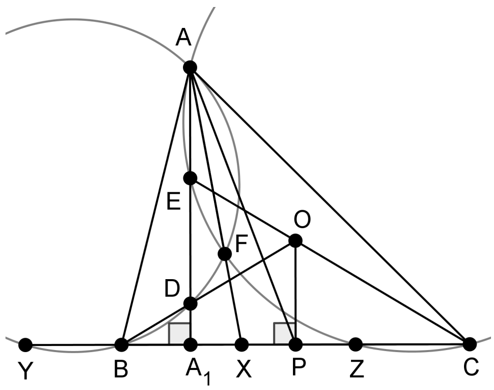
Example 13.1 (Macedonia MO 2017, Stefan Lozanovski)**.** Let O be the circumcenter of the acute triangle ABC ($\overline{AB} < \overline{AC}$). Let A_1 and P be the feet of the perpendiculars from A and O to BC, respectively. The lines BO and CO intersect AA_1 in D and E, respectively. Let F be the second intersection point of (ABD) and (ACE). Prove that the angle bisector od $\angle FAP$ passes through the incenter of $\triangle ABC$.

Proof. We need to prove that $\angle BAF = \angle CAP$. Since OP is perpendicular to BC and O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

Proof 1. We will use Property 13.1 to prove that AF is symmedian. Let the line AF intersect the side BC at X and let the circumcircles of $\triangle ABD$ and $\triangle ACE$ meet the line BC again at Y and Z, respectively. Then, by the Intersecting Secants Theorem, we have

$$\overline{XB} \cdot \overline{XY} = \overline{XF} \cdot \overline{XA} = \overline{XZ} \cdot \overline{XC}$$

$$\frac{\overline{XB}}{\overline{XC}} = \frac{\overline{XZ}}{\overline{XY}} = \frac{\overline{XB} + \overline{XZ}}{\overline{XC} + \overline{XY}} = \frac{\overline{BZ}}{\overline{CY}} \tag{1}$$



Now, let's use the fact that the point E is defined as the intersection of the altitude and the circumradius.

$$\angle ACE \equiv \angle ACO = \frac{1}{2}(180^\circ - \angle AOC) = \frac{1}{2}(180^\circ - 2\angle ABC) =$$
$$= 90^\circ - \angle ABC \equiv 90^\circ - \angle ABA_1 = \angle BAA_1 \equiv \angle BAE$$

Therefore, BA is tangent to (ACE). Similarly, CA is tangent to (ABD). Now, by the Secant-Tangent Theorem, we have

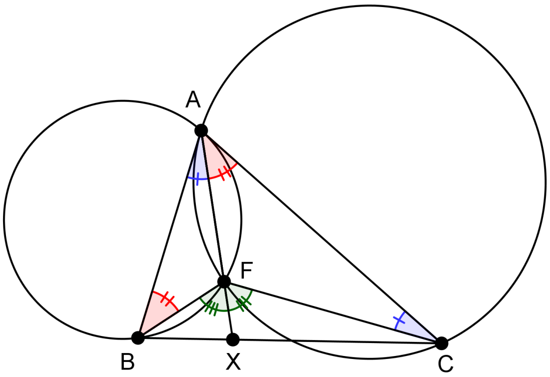
$$\overline{BA}^2 = \overline{BZ} \cdot \overline{BC}$$
$$\overline{CA}^2 = \overline{CB} \cdot \overline{CY}$$

By dividing these equations and using (1), we get

$$\frac{\overline{BA}^2}{\overline{CA}^2} = \frac{\overline{BZ}}{\overline{CY}} = \frac{\overline{XB}}{\overline{XC}} \qquad \blacksquare$$

Proof 2. As in Proof 1, we will use Property 13.1 to prove that AF is symmedian. In Proof 1, we also proved that BA is tangent to $(ACE) \equiv (ACF)$. Therefore, $\angle BAF = \angle ACF$. Similarly, CA is tangent to ABF and therefore $\angle CAF = \angle ABF$. Thus, by the criterion AA, $\triangle BAF \sim \triangle ACF$ which gives

$$\frac{\overline{BF}}{\overline{CF}} = \frac{\frac{\overline{BF}}{\overline{AF}}}{\frac{\overline{CF}}{\overline{AF}}} = \frac{\frac{\overline{BA}}{\overline{AC}}}{\frac{\overline{AC}}{\overline{AB}}} = \frac{\overline{AB}^2}{\overline{AC}^2}$$



Also, $\angle BFX = 180° - \angle BFA = 180° - \angle AFC = \angle CFX$, so FX is an angle bisector in $\triangle BFC$ and

$$\frac{\overline{BF}}{\overline{CF}} = \frac{\overline{BX}}{\overline{CX}}$$

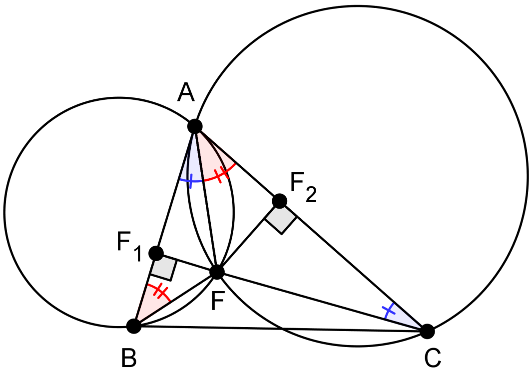
Finally, we get that

$$\frac{\overline{BX}}{\overline{CX}} = \frac{\overline{AB}^2}{\overline{AC}^2}$$ ■

Proof 3. Same as in Proof 2, we get that $\triangle BAF \sim \triangle ACF$. Let F_1 and F_2 be the feet of the perpendiculars from F to AB and AC, respectively. Then, from the similarity, we get

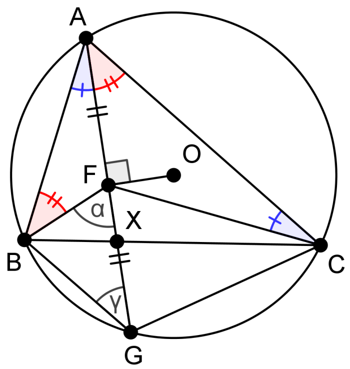
$$\frac{\overline{FF_1}}{\overline{FF_2}} = \frac{\overline{AB}}{\overline{AC}}$$

which, by Property 13.4, means that F lies on the $A-$symmedian. ■



Proof 4. Same as in Proof 2, we get that $\triangle BAF \sim \triangle ACF$ and therefore

$$\frac{\overline{BA}}{\overline{BF}} = \frac{\overline{AC}}{\overline{AF}} \tag{1}$$



Let AX intersect (ABC) again at G. Then,

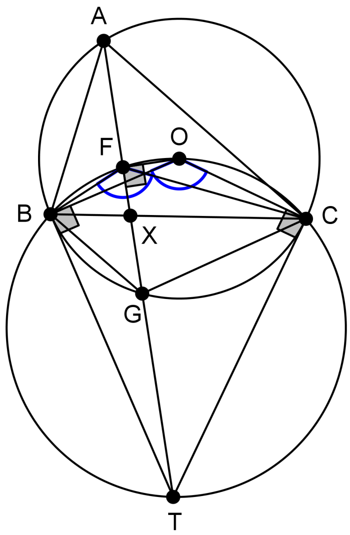
$$\angle BFG = 180^\circ - \angle BFA = \angle FBA + \angle FAB = \angle FAC + \angle FAB = \alpha$$

$$\angle BGF \equiv \angle BGA = \angle BCA = \gamma$$

$$\therefore \triangle ABC \sim \triangle FBG$$

$$\therefore \frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AC}}{\overline{FG}} \tag{2}$$

From (1) and (2), we get that $\overline{AF} = \overline{FG}$, i.e. F is the midpoint of the chord AG. Since O is the circumcenter, we get $OF \perp AG$, i.e. $\angle OFG = 90^\circ$ (3)



Now, let's draw the tangents to the circumcircle at B and C, and let them intersect at T. The quadrilateral $OBTC$ is cyclic with diameter OT. Earlier in this solution, we proved that $\angle BFG = \alpha$. Similarly, $\angle CFG = \alpha$.

$$\therefore \angle BFC = \angle BFG + \angle CFG = 2\alpha = \angle BOC$$

$$\therefore F \in (BOCT)$$

$$\therefore \angle OFT = 90^\circ$$

$$\therefore \angle OFT \overset{(3)}{=} \angle OFG$$

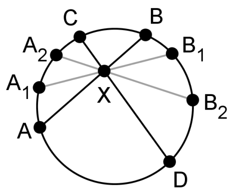
$$T \in FG \equiv AF$$

Finally, since AF passes through the intersection of the tangents at B and C, by Property 13.3, AF is symmedian. ■

Related problems: 122, 126, 132, 139, 189, 193 and 195.

Chapter 14

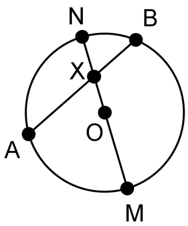
Power of a Point



Let AB and CD be two intersecting chords in a circle and let their intersection be X. By the Intersecting Chords Theorem, $\overline{AX} \cdot \overline{XB} = \overline{CX} \cdot \overline{XD}$. This means that for a fixed point X in the fixed circle $\omega(O,\, r)$, the product $\overline{AX} \cdot \overline{XB}$ will be constant and will not depend on the choice of the chord A_iB_i which passes through X, i.e.

$$\overline{AX} \cdot \overline{XB} = \overline{A_1X} \cdot \overline{XB_1} = \overline{A_2X} \cdot \overline{XB_2} = const.$$

So, this product must depend on the position of X (relative to ω) and on ω itself.



Well, let's try to express this product as a function of the known elements, i.e. the radius of the circle and the distance from the center of the circle to the point X. Let's draw a diameter through X (in order to include the center in all of this) and let M and N be its endpoints. Then, as previously proved,

$$\overline{AX} \cdot \overline{XB} = \overline{MX} \cdot \overline{XN} = (\overline{MO} + \overline{OX})(\overline{ON} - \overline{OX}) =$$
$$= (r + \overline{OX})(r - \overline{OX}) = r^2 - \overline{OX}^2.$$

This is, in fact, the absolute value (since X is inside the circle) of what we will call the power of X with respect to ω.

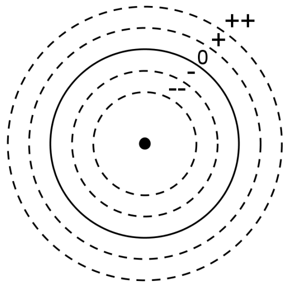
We will define the *power of the point* X with respect to the circle $\omega(O,\, r)$ as a real number which reflects the relative distance of the point X to the circle ω:

$$pow(X,\, \omega) = \overline{OX}^2 - r^2.$$

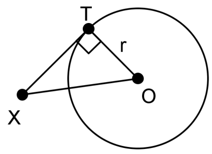
Consequently, we can conclude the following property:

Property 14.1. Points that are on equal distances from the center have equal powers with respect to the circle.

By the definition, it also means that the points inside the circle (for which $0 \leq \overline{OX} < r$) will have negative power, the points on the circle (for which $\overline{OX} = r$) will have zero power and the points outside the circle (for which $\overline{OX} > r$) will have positive power with respect to the circle.



Property 14.2. If a point X is outside the circle $\omega(O, r)$, then the power of the point equals the square of the length of the tangent segment from X to the tangent point T.

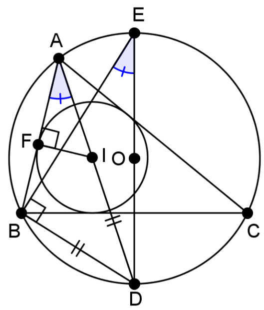


Proof. By the definition of power of a point and using the Pythagorean Theorem:

$$pow(X, \omega) = \overline{OX}^2 - r^2 = \overline{OX}^2 - \overline{OT}^2 = \overline{XT}^2 \quad \blacksquare$$

Example 14.1 (Euler's Theorem in Geometry)**.** Let O and I be the circumcenter and incenter of $\triangle ABC$, respectively. Let R and r be the circumradius and inradius, respectively. Prove that

$$\overline{OI}^2 = R(R - 2r)$$



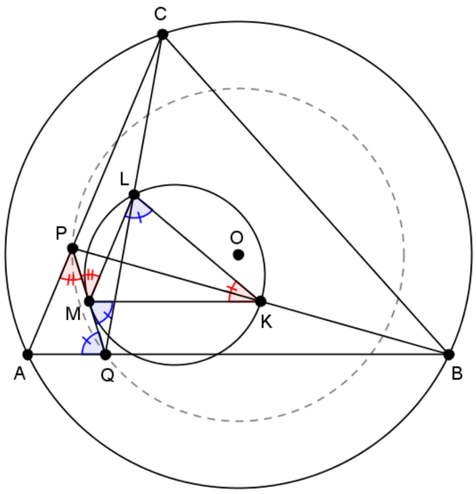
Proof. Let D be the second intersection of AI with the circumcircle ω of $\triangle ABC$. We need to prove that $2Rr = R^2 - \overline{OI}^2 = |pow(I, \omega)| = \overline{AI} \cdot \overline{ID}$. By Property 7.2, we know that $\overline{DI} = \overline{DB}$. So, we need to find similar triangles that contain the sides AI, BD, r and $2R$. Let E be the diametrically opposite point of D on ω; Thus $\overline{ED} = 2R$ and $\angle EBD = 90°$. Let F be tangent point of the incircle with the side AB; Thus $\overline{IF} = r$ and $\angle IFA = 90°$. Since $\angle FAI \equiv \angle BAD = \angle BED$, the right triangles $\triangle AIF$ and $\triangle EDB$ are similar. Therefore,

$$\frac{\overline{AI}}{\overline{IF}} = \frac{\overline{ED}}{\overline{DB}} \tag{*}$$

$$\overline{AI} \cdot \overline{ID} = \overline{AI} \cdot \overline{DB} \overset{(*)}{=} \overline{ED} \cdot \overline{IF} = 2Rr \quad \blacksquare$$

Remark. From this theorem, we can derive the *Euler inequality*. Since $\overline{OI}^2$ is non-negative and R is always positive, we can conclude that $R - 2r$ is non-negative, i.e. $R \geq 2r$. Equality holds iff $O \equiv I$, i.e. when $\triangle ABC$ is equilateral.

Example 14.2 (IMO 2009/2)**.** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $\overline{OP} = \overline{OQ}$.



Proof. We need to prove that P and Q are on the same distance from the circumcenter O, so by Property 14.1, we need to prove that their power with respect to the circumcircle (ABC) are equal, i.e. we need to prove that

$$\overline{AP} \cdot \overline{PC} = \overline{AQ} \cdot \overline{QB}.$$

Since MK and ML are midsegments in $\triangle QBP$ and $\triangle PCQ$, we have

$$MK \parallel QB \quad \text{and} \quad ML \parallel PC \tag{1}$$

$$\overline{MK} = \frac{1}{2} \cdot \overline{QB} \quad \text{and} \quad \overline{ML} = \frac{1}{2} \cdot \overline{PC} \tag{2}$$

Since Γ is tangent to PQ at M, and using (1), we get

$$\angle KLM = \angle KMQ = \angle MQA \equiv \angle PQA$$

$$\angle LKM = \angle LMP = \angle MPA \equiv \angle QPA$$

Therefore, $\triangle APQ \sim \triangle MKL$.

$$\therefore \frac{\overline{AP}}{\overline{AQ}} = \frac{\overline{MK}}{\overline{ML}} \stackrel{(2)}{=} \frac{\overline{QB}}{\overline{PC}}, \text{ i.e. } \overline{AP} \cdot \overline{PC} = \overline{AQ} \cdot \overline{QB} \qquad \blacksquare$$

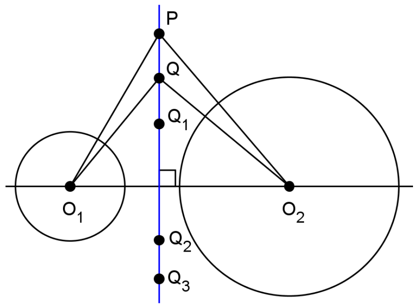
Before continuing, let's mention that sometimes it's useful to consider a point as a degenerate circle, i.e. a circle with radius zero. Then, the power of a point P with respect to the degenerate circle $\omega(A, 0)$ is

$$pow(P, \omega) = \overline{AP}^2 - r^2 = \overline{AP}^2$$

If it is unclear to you what this means, wait just a bit; We will use this in Example 14.3.

14.1 Radical Axis

We learned about the power of a point with respect to a circle. Now, let's find the locus of the points that have equal power with respect to two given circles $\omega_1(O_1, r_1)$ and $\omega_2(O_2, r_2)$. Let P be a point that satisfies this condition. Then,



by the definition of power of a point,

$$\overline{PO_1}^2 - r_1^2 = \overline{PO_2}^2 - r_2^2.$$

Let Q be another point that satisfies the condition. Similarly, we have

$$\overline{QO_1}^2 - r_1^2 = \overline{QO_2}^2 - r_2^2.$$

From the two equations above, we get:

$$\overline{PO_1}^2 - \overline{PO_2}^2 = r_1^2 - r_2^2 = \overline{QO_1}^2 - \overline{QO_2}^2,$$

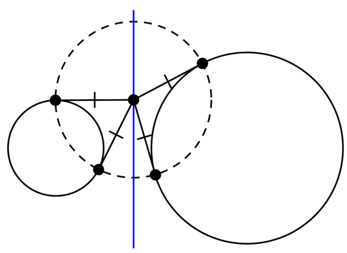
which, by Property 4.3 means that $PQ \perp O_1O_2$. But this will also be true for all other points $Q_1, Q_2, \ldots$ that have same power with respect to both circles, i.e. $PQ_1 \perp O_1O_2$, $PQ_2 \perp O_1O_2$, $\ldots$, which means that the set of all such points is a straight line perpendicular to O_1O_2.

The radical axis of two circles is a line that is the locus of all points that have equal powers with respect to both circles.

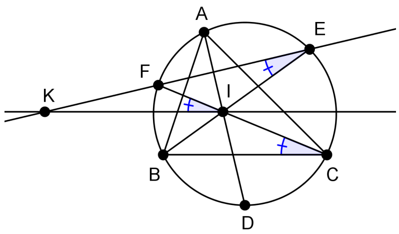
Property 14.3. The radical axis of two circles is perpendicular to the line connecting the centers of the circles.

As a consequence, the radical axis of two intersecting circles will be the line that passes through their intersection points, because those points have zero power with respect to both circles. The radical axis of two tangent circles will be their common tangent through their tangent point, because it is perpendicular to the line connecting the centers and because the tangent point has zero power with respect to both circles.

Let's recall that if a point is outside the circle, the power of the point with respect to the circle equals the square of the length of the tangent segment from the point to the circle. Hence, the tangent segments from such point Q_i to both circles are of equal length, which means that each point on the radical axis is a center of a circle that intersects both given circles orthogonally.



Example 14.3 (BMO 2015)**.** Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI and CI intersect ω for the second time at points D, E and F, respectively. The parallel lines from I to the sides BC, AC and AB intersect EF, DF and DE at points K, L and M, respectively. Prove that the points K, L and M are collinear.



Proof. Because of the parallel lines KI and BC,

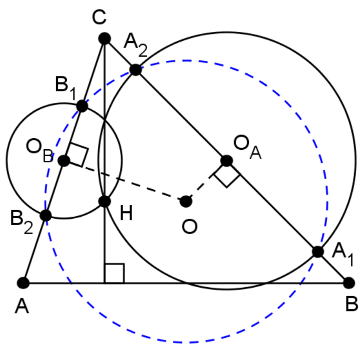
$$\angle KIF = \angle BCI \equiv \angle BCF \stackrel{\omega}{=} \angle BEF \equiv \angle IEK$$

In addition to this, $\angle IKE \equiv \angle IKF$ is a common angle for the triangles $\triangle KIF$ and $\triangle KEI$, so the triangles are similar and therefore

$$\frac{\overline{KI}}{\overline{KF}} = \frac{\overline{KE}}{\overline{KI}}, \text{ i.e. } \overline{KI}^2 = \overline{KF} \cdot \overline{KE}$$

The left hand side is the power of the point K with respect to the degenerate circle I and the right hand side is the power of the point K with respect to the circle (EFD). Therefore, K lies on the radical axis r of I and (EFD). Similarly, L and M also lie on r, so they are collinear. ■

Example 14.4 (IMO 2008/1)**.** Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.



Proof. A precise drawing may give us a hint about this problem. If we draw the figure correctly, we will see that the second intersection of the circles Γ_A and Γ_B lies on the altitude CH. So, we firstly need to prove that this is indeed true, i.e. we need to prove that CH is the radical axis of Γ_A and Γ_B and then use this fact to solve the problem.

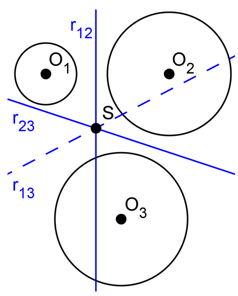
Let O_A and O_B be the centers of Γ_A and Γ_B (i.e. the midpoints of BC and CA), respectively. Since O_AO_B is a midsegment in $\triangle ABC$, $O_AO_B \parallel AB$. Since CH is altitude in $\triangle ABC$, $AB \perp CH$. Therefore, $O_AO_B \perp CH$. By Property 14.3 and recalling that $H \in \Gamma_A$ and $H \in \Gamma_B$, we can conclude that CH is the radical axis of Γ_A and Γ_B.

Now, using the fact that any point on the radical axis has equal power with respect to both circles, we get that

$$\overline{CA_1} \cdot \overline{CA_2} = \overline{CB_1} \cdot \overline{CB_2}.$$

Therefore, $A_1A_2B_1B_2$ is a cyclic quadrilateral. The center of $(A_1A_2B_1B_2)$ can be found as the intersection of the side bisectors of A_1A_2 and B_1B_2. But, since the midpoint of BC coincides with the midpoint of A_1A_2 and the midpoint of CA coincides with the midpoint of B_1B_2, then the side bisectors of A_1A_2 and B_1B_2 intersect at the circumcenter O of $\triangle ABC$. Similarly, $(B_1B_2C_1C_2)$ is a circle centered at O. Therefore, the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic. ■

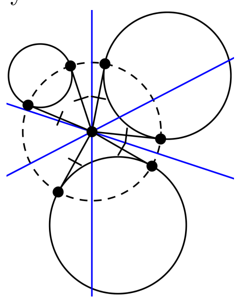
14.2 Radical Center



Let's find the locus of the points that have equal power with respect to three circles ω_1, ω_2 and ω_3, whose centers are not collinear. By definition, the set of points that satisfy $pow(X, \omega_1) = pow(X, \omega_2)$ is the radical axis of ω_1 and ω_2 and the set of points that satisfy $pow(X, \omega_2) = pow(X, \omega_3)$ is the radical axis of ω_2 and ω_3. These axes are not parallel (because the centers of the circles are not collinear), so let their intersection be S. By transitivity, we get that for this point S, the following is true $pow(S, \omega_1) = pow(S, \omega_3)$, which means that the radical axis of ω_1 and ω_3 also passes through S. So, S is the only point that has equal power to all three circles and it is called the *radical center* of the three circles.

If the centers of ω_1, ω_2 and ω_3 are collinear, then their pairwise radical axes are all parallel (since they are perpendicular to the line connecting the centers). Here, we have two cases: either the pairwise radical axes are three different lines, in which case there is no point that has equal power with respect to the three circles; or the three circles have a common chord (i.e. they are *coaxial*), in which case all points on this common axis have equal power with respect to the three circles.

Note that if the radical center lies outside of all three circles, then the tangent segments from it to all three circles will be of equal length. So, the radical center is the center of the unique circle (called the *radical circle*) that intersects the three given circles orthogonally.



Geometric construction of radical axis

Now, let's see how we can geometrically construct the radical axis of two non-concentric circles.

i) $\omega_1 \cap \omega_2 = \{A, B\}$
The points A and B lie on both circles, so they both have zero power to both circles. Since we know that the radical axis is a line, we can construct it by drawing the line through the points A and B.

ii) $\omega_1 \cap \omega_2 = \{T\}$
As discussed in the previous case, the point T lies on the radical axis. Since we proved that the radical axis is a line perpendicular to the line joining the centers, we can construct it easily as the common tangent of the circles through T.

iii) $\omega_1 \cap \omega_2 = \emptyset$
Let's draw another circle ω_3 that intersects both ω_1 and ω_2. Let's construct the radical axes of ω_1 and ω_3, $r_{1,3}$, and ω_2 and ω_3, $r_{2,3}$. The intersection of $r_{1,3}$ and $r_{2,3}$, S, is the radical center of the three circles, so it must lie on the radical axis $r_{1,2}$ that we are trying to construct. Now, we can continue in two different ways: we can either do the same thing with another circle that intersect ω_1 and ω_2, thus finding another point that lies on $r_{1,2}$; or we can use the fact the radical axis $r_{1,2}$ is perpendicular to the line joining the centers, O_1O_2.

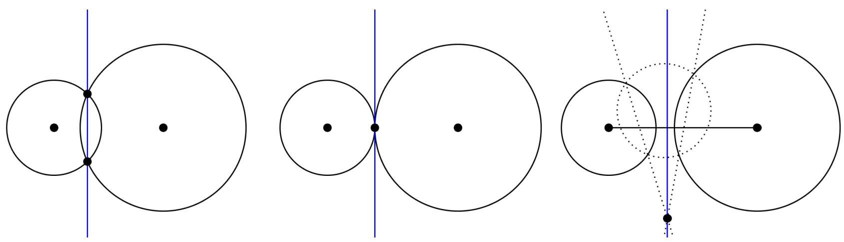


Figure 14.1: Radical axis of two circles

Related problems: 40, 102, 109, 111, 113, 117, 171, 183 and 186.

Chapter 15

Collinearity II

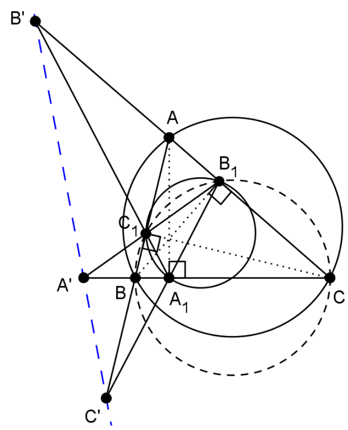
15.1 Radical Axis

In some problems, we can prove that three points are collinear if we show that they all lie on the Radical Axis of some two circles.

Example 15.1.1 (Orthic axis). Let AA_1, BB_1 and CC_1 be the altitudes in $\triangle ABC$. Let A' be the intersection of the lines BC and B_1C_1 and similarly define the points B' and C'. Prove that A', B' and C' lie on a line.

Proof. Since $\angle BB_1C = 90° = \angle BC_1C$, the quadrilateral BCB_1C_1 is cyclic. Therefore, by the intersecting secant theorem, we have

$$\overline{A'B} \cdot \overline{A'C} = \overline{A'C_1} \cdot \overline{A'B_1}.$$



The left-hand side is in fact the power of the point A' to the circle (ABC) and the right-hand side is the power of the point A' to the circle $(A_1B_1C_1)$. Since it has same power with respect to both circles, then it must lie on the radical axis of those circle. Similarly, B' and C' also lie on the radical axis of (ABC) and $(A_1B_1C_1)$, so A', B' and C' are collinear. ■

15.2 Menelaus' Theorem

Property 15.2.1 (Menelaus' Theorem)**.** Let ABC be a triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, such that odd number of them (one or three) are on the extensions of the sides. The points D, E and F are collinear if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

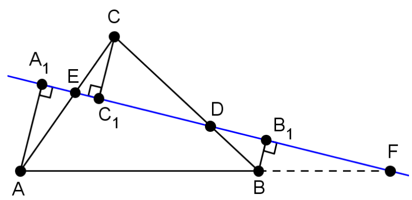
Remark. An easy way to remember how to write these ratios is the following. If we have a triangle $\triangle XYZ$ and the points $M \in XY$, $N \in YZ$ and $P \in ZX$ lie on its sides, then we will write the ratios as follows:
Firstly we are going to write its sides in a cyclic manner, like this

$$\frac{\overline{X\quad}}{\overline{\quad Y}} \cdot \frac{\overline{Y\quad}}{\overline{\quad Z}} \cdot \frac{\overline{Z\quad}}{\overline{\quad X}}$$

and then we will just add each point in the numerator and denominator in the fraction of the corresponding side, like this

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YN}}{\overline{NZ}} \cdot \frac{\overline{ZP}}{\overline{PX}}$$

Proof. Let D, E and F be collinear and let the line defined by them be p. Let A_1, B_1 and C_1 be the feet of the perpendiculars from A, B and C, respectively, to the line p.



$$\triangle AA_1F \sim \triangle BB_1F \quad (\because \angle AA_1F = 90^\circ = \angle BB_1F,\ \angle AFA_1 \equiv \angle BFB_1)$$

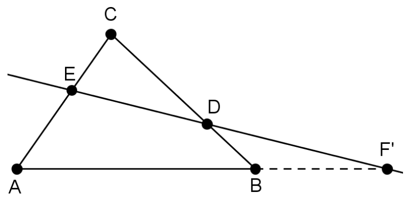
$$\therefore \frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AA_1}}{\overline{BB_1}}$$

$$\text{Similarly, } \frac{\overline{BD}}{\overline{DC}} = \frac{\overline{BB_1}}{\overline{CC_1}} \text{ and } \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CC_1}}{\overline{AA_1}}.$$

$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AA_1}}{\overline{BB_1}} \cdot \frac{\overline{BB_1}}{\overline{CC_1}} \cdot \frac{\overline{CC_1}}{\overline{AA_1}} = 1 \quad \square$$

Now, let's prove the other direction. Let

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



We will discuss the case when two of the points are on the sides of the triangle and one on the extension of the other side. The other case, when all three points are on the extensions of the sides is analogous. WLOG, let D and E be on the sides BC and CA, respectively and F be on the extension of the side AB. We should prove that the points D, E and F are collinear. Let the line DE intersect the line AB at F' (note that F' cannot lie between A and B). Because the points D, E and F' are collinear, we can use the direction of the Menelaus' Theorem that we just proved. So,

$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Combining with the given condition, we get

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AF'}}{\overline{F'B}}.$$

Since $\overline{AF} - \overline{FB} = \overline{AB} = \overline{AF'} - \overline{F'B}$, by subtracting 1 from both sides in the above equation, we get

$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AB}}{\overline{F'B}}.$$

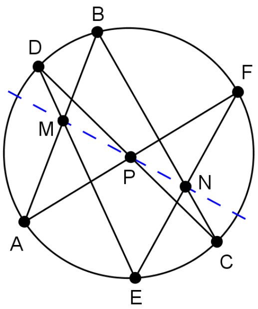
We conclude that $\overline{FB} = \overline{F'B}$. Because both F and F' are on the extension of the side AB, we get that $F \equiv F'$, i.e. the points D, E and F are collinear. $\blacksquare$

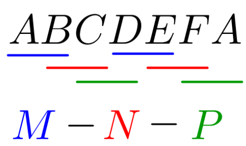
We will see how this theorem can be used in both directions, while proving the next theorem.

Related problems: (Menelaus) 161, 177, 178, 184 and 205.

15.3 Pascal's Theorem

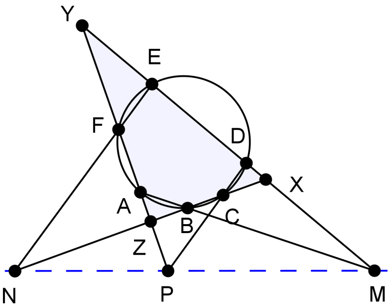
Property 15.3.1 (Pascal's Theorem)**.** Let A, B, C, D, E and F be points on a circle (not necessarily in cyclic order). Let $M = AB \cap DE$, $N = BC \cap EF$ and $P = CD \cap FA$. Then M, N and P are collinear.





Remark. An easy way to remember these intersections is the following: Take two consecutive letters for a line, skip one letter, and take two more letters for the second line. Their intersection is the first of the three collinear points. Then shift to the right and repeat two times.

Proof. Let $X = BC \cap DE$, $Y = DE \cap FA$ and $Z = FA \cap BC$.



If we use Menelaus' Theorem three times on $\triangle XYZ$, firstly with the collinear points $A - B - M$, then with the collinear points $P - C - D$ and finally with the collinear points $F - N - E$, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YA}}{\overline{AZ}} \cdot \frac{\overline{ZB}}{\overline{BX}} = 1$$

$$\frac{\overline{XD}}{\overline{DY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZC}}{\overline{CX}} = 1$$

$$\frac{\overline{XE}}{\overline{EY}} \cdot \frac{\overline{YF}}{\overline{FZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} = 1$$

By multiplying these 3 equations and reordering the members, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} \cdot \frac{(\overline{YA} \cdot \overline{YF}) \cdot (\overline{ZB} \cdot \overline{ZC}) \cdot (\overline{XD} \cdot \overline{XE})}{(\overline{AZ} \cdot \overline{FZ}) \cdot (\overline{BX} \cdot \overline{CX}) \cdot (\overline{DY} \cdot \overline{EY})} = 1.$$

From the Intersecting Secants Theorem for the points X, Y and Z we get:

$$\overline{XD} \cdot \overline{XE} = \overline{XC} \cdot \overline{XB}$$

$$\overline{YF} \cdot \overline{YA} = \overline{YE} \cdot \overline{YD}$$

$$\overline{ZB} \cdot \overline{ZC} = \overline{ZA} \cdot \overline{ZF}$$

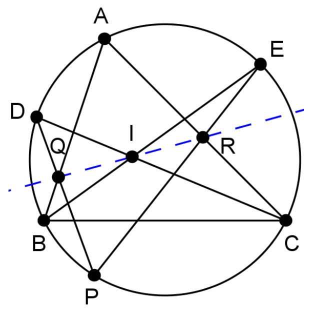
From the previous four equations, we get:

$$\frac{\overline{XM}}{\overline{MY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZN}}{\overline{NX}} = 1,$$

which by Menelaus' Theorem means that M, N and P are collinear. ■

Here is an example to show how the Pascal's Theorem can be used in a problem.

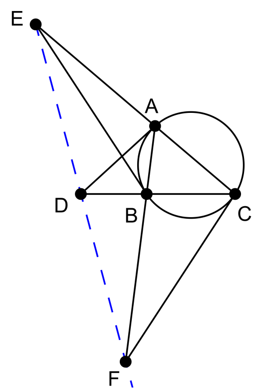
Example 15.3.1. Let D and E be the midpoints of the minor arcs $\widehat{AB}$ and $\widehat{AC}$ on the circumcircle of $\triangle ABC$, respectively. Let P be on the minor arc $\widehat{BC}$, $Q = PD \cap AB$ and $R = PE \cap AC$. Prove that the line QR passes through the incenter I of $\triangle ABC$.



Proof. Since D is the midpoint of the arc $\widehat{AB}$, CD is the angle bisector of $\angle BCA$. Similarly, BE is the angle bisector of $\angle ABC$. Therefore, $CD \cap BE = I$. Now, we apply Pascal's Theorem to the points C, D, P, E, B and A and we get that the points $CD \cap EB = I$, $DP \cap BA = Q$ and $PE \cap AC = R$ are collinear. ■

We remark that there are limiting cases of Pascal's Theorem. For example, we may move A to approach B. In the limit, A and B will coincide and the line AB will become the tangent line at B. Here is an example to show how this works.

Example 15.3.2. Let ω be the circumcircle of $\triangle ABC$. Let the tangent lines to ω at A, B and C intersect the lines BC, CA and AB at points D, E and F, respectively. Prove that the points D, E and F are collinear.

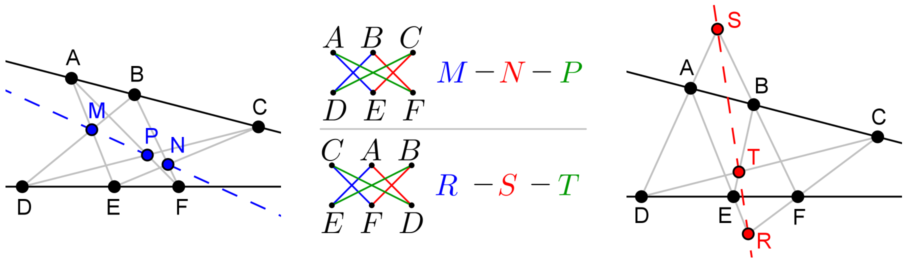


Proof. Let's apply the Pascal's Theorem to the points A, A, B, B, C and C. We get that the points $AA \cap BC = D$, $AB \cap CC = F$ and $BB \cap CA = E$ are collinear. ■

Related problem: (Pascal) 180.

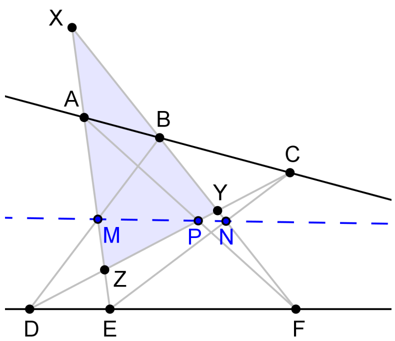
15.4 Pappus' Theorem

Property 15.4.1 (Pappus' Hexagon Theorem)**.** Let A, B, C be three collinear points (not necessarily in this order) and D, E, F be another three collinear points (not necessarily in this order). Let $M = AE \cap BD$, $N = BF \cap CE$ and $P = AF \cap CD$. Then, the points M, N and P are collinear.



Proof. We'll divide the proof in two cases, based on whether the lines AE and BF intersect or are parallel. If they intersect, let $AE \cap BF = X$. Let $Y = BF \cap CD$ and $Z = AE \cap CD$. Since M, N and P lie on the sidelines of $\triangle XYZ$, we can use Menelaus' Theorem to try to prove that they are collinear, i.e. we need to prove that:

$$\frac{\overline{XN}}{\overline{NY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZM}}{\overline{MX}} = 1.$$



Now, we try to find each of these three ratios in a Menelaus ratio for other lines that intersect the sidelines of $\triangle XYZ$. In order to achieve this, we use Menelaus' Theorem three times for $\triangle XYZ$ and for the lines $C-E-N$, $P-A-F$ and $D-M-B$, respectively. We get:

$$\frac{\overline{XN}}{\overline{NY}} \cdot \frac{\overline{YC}}{\overline{CZ}} \cdot \frac{\overline{ZE}}{\overline{EX}} = 1\,, \quad \frac{\overline{XF}}{\overline{FY}} \cdot \frac{\overline{YP}}{\overline{PZ}} \cdot \frac{\overline{ZA}}{\overline{AX}} = 1 \quad \text{and} \quad \frac{\overline{XB}}{\overline{BY}} \cdot \frac{\overline{YD}}{\overline{DZ}} \cdot \frac{\overline{ZM}}{\overline{MX}} = 1.$$

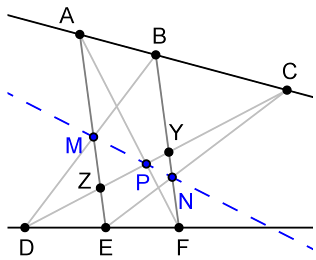
But, we see that nothing cancels out, so we need to find other equalities that use these line segments. In order to achieve this, we use Menelaus' Theorem two more times for $\triangle XYZ$, but this time for the points $C-A-B$ and $D-E-F$. We get:

$$1 = \frac{\overline{XB}}{\overline{BY}} \cdot \frac{\overline{YC}}{\overline{CZ}} \cdot \frac{\overline{ZA}}{\overline{AX}} \quad \text{and} \quad 1 = \frac{\overline{XF}}{\overline{FY}} \cdot \frac{\overline{YD}}{\overline{DZ}} \cdot \frac{\overline{ZE}}{\overline{EX}}.$$

We multiply these 5 equalities side by side, and we see that each of the 6 fractions on the RHS cancels out with a fraction from the LHS. We are left with what we needed to prove $\square$.

Now, let's see what happens if the point X doesn't exist, i.e. if $AE \parallel BF$. We observe that the point X appears exactly twice in each of the five equalities above, once in the numerator and once in the denominator. We try to find "analogous" equalities that don't use the lengths that contain X. We can actually achieve this, by using the parallel lines to find pairs of similar triangles. By using Property 2.1, by the criterion AA, we get $\triangle CYN \sim \triangle CZE$.

$$\therefore \frac{\overline{CY}}{\overline{YN}} = \frac{\overline{CZ}}{\overline{ZE}}, \text{ i.e. } \frac{\overline{CY}}{\overline{YN}} \cdot \frac{\overline{ZE}}{\overline{CZ}} = 1$$



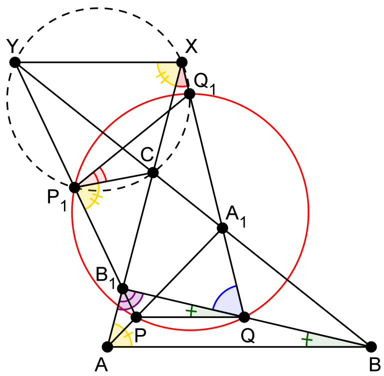
Similarly, we get $\triangle PYF \sim \triangle PZA$, $\triangle DYB \sim \triangle DZM$, $\triangle YCB \sim \triangle ZCA$ and $\triangle YDF \sim \triangle ZDE$. By multiplying their analogous equalities, we get:

$$\frac{\overline{CY}}{\overline{YN}} \cdot \frac{\overline{ZE}}{\overline{CZ}} \cdot \frac{\overline{PY}}{\overline{YF}} \cdot \frac{\overline{ZA}}{\overline{PZ}} \cdot \frac{\overline{DY}}{\overline{YB}} \cdot \frac{\overline{ZM}}{\overline{DZ}} \cdot \frac{\overline{YB}}{\overline{YC}} \cdot \frac{\overline{ZC}}{\overline{ZA}} \cdot \frac{\overline{YF}}{\overline{YD}} \cdot \frac{\overline{ZD}}{\overline{ZE}} = 1$$

$$\therefore \frac{\overline{PY}}{\overline{YN}} \cdot \frac{\overline{ZM}}{\overline{PZ}} = 1, \text{ i.e. } \frac{\overline{PY}}{\overline{YN}} = \frac{\overline{PZ}}{\overline{ZM}}$$

Since $\angle PYN = \angle PZM$, by SAS we get $\triangle PYN \sim \triangle PZM$ and therefore $\angle YPN = \angle ZPM$. Since $Y - P - Z$ are collinear, then so must be $N - P - M$. ■

Example 15.4.1 (IMO 2019/2)**.** In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1, and Q_1 are concyclic.



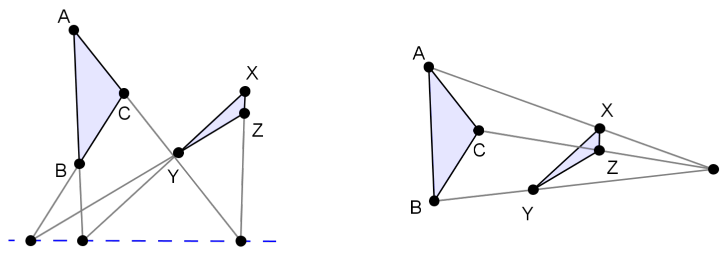
Proof. We need to prove that PQQ_1P_1 is cyclic, i.e. $\angle PQQ_1 + \angle PP_1Q_1 = 180°$, i.e. $\angle PQB_1 + \angle B_1QQ_1 + \angle PP_1C + \angle CP_1Q_1 = 180°$. But, we know that $\angle PQB_1 = \angle ABB_1$ and $\angle PP_1C = \angle BAB_1$ and we also know that $\angle ABB_1 + \angle BAB_1 = 180° - \angle AB_1B$, so we need to prove that $\angle B_1QQ_1 + \angle CP_1Q_1 = \angle AB_1B$. We see that $\angle AB_1B$ is already an exterior angle in a triangle where one of the internal angles is exactly $\angle B_1QQ_1$. That's why we are motivated to define $X = QQ_1 \cap B_1C$. Since $\angle AB_1B = \angle B_1QX + \angle B_1XQ$, we need to prove that $\angle CP_1Q_1 = \angle B_1XQ \equiv \angle CXQ_1$, i.e. we need to prove that CP_1XQ_1 is cyclic. By similar reasoning, if $Y = PP_1 \cap A_1C$, we need to prove that CP_1YQ_1 is cyclic. But that would mean that C, P_1, Q_1, X, Y all lie on one circle, which is equivalent to $\angle CXY = \angle CP_1P = \angle CAB$, i.e. $XY \parallel AB$. So, if we prove that $XY \parallel AB$, we are done.

If we consider the two sets of collinear points $A - A_1 - P$ and $Q - B_1 - B$, by Pappus' Hexagon Theorem, we get that $AB_1 \cap A_1Q = X$, $A_1B \cap PB_1 = Y$ and $PQ \cap BA$ are collinear, i.e. XY, PQ, AB are concurrent. But, $PQ \parallel AB$, i.e. they intersect at "the point at infinity", so in order for XY to be "concurrent" with them, they must be all parallel, i.e. $XY \parallel AB$. ∎

15.5 Desargues' Theorem

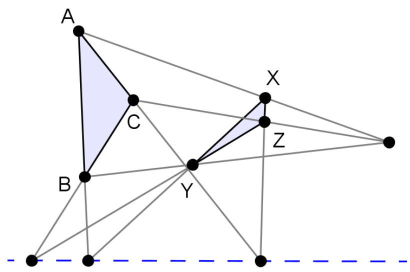
Two triangles $\triangle ABC$ and $\triangle XYZ$ are *perspective from a line* if the points $AB \cap XY$, $BC \cap YZ$ and $CA \cap ZX$ are collinear.

Two triangles $\triangle ABC$ and $\triangle XYZ$ are *perspective from a point* if the lines AX, BY and CZ are concurrent.



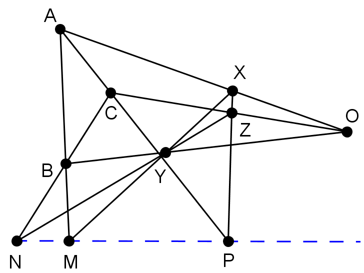
(a) Perspective from a line. (b) Perspective from a point.

Property 15.5.1 (Desargues' Theorem)**.** Two triangles are perspective from a line if and only if they are perspective from a point.



Proof. Let $\triangle ABC$ and $\triangle XYZ$ be perspective from a point, i.e. AX, BY and CZ are concurrent, and let the point of concurrence be O. Let $AB \cap XY = M$, $BC \cap YZ = N$ and $CA \cap ZX = P$.

We firstly apply the Menelaus' Theorem to $\triangle OAB$ and the points $M-Y-X$, then to $\triangle OBC$ and $N-Z-Y$, and finally to $\triangle OCA$ and $P-X-Z$:



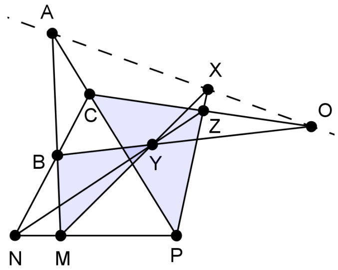
$$\frac{\overline{OX}}{\overline{XA}} \cdot \frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BY}}{\overline{YO}} = 1$$
$$\frac{\overline{OY}}{\overline{YB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CZ}}{\overline{ZO}} = 1$$
$$\frac{\overline{OZ}}{\overline{ZC}} \cdot \frac{\overline{CP}}{\overline{PA}} \cdot \frac{\overline{AX}}{\overline{XO}} = 1$$

By multiplying these three equations, we get:

$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CP}}{\overline{PA}} = 1,$$

which by the Menelaus' Theorem for $\triangle ABC$, means that the points M, N and P are collinear, i.e. $\triangle ABC$ and $\triangle XYZ$ are perspective from a line. □

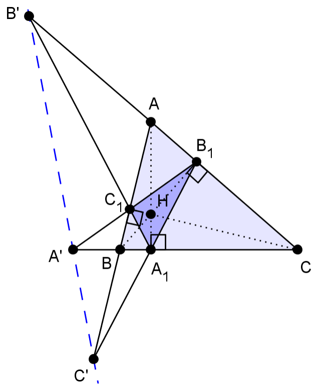
Now, let's prove the other direction. Let $\triangle ABC$ and $\triangle XYZ$ be perspective from a line, i.e. the points $M = AB \cap XY$, $N = BC \cap YZ$ and $P = CA \cap ZX$ are collinear. Let $O = BY \cap CZ$. We should prove that AX passes through O.



Let's take a look at $\triangle PCZ$ and $\triangle MBY$. The lines PM, CB and ZY are concurrent at N, so the triangles are perspective from a point. By the direction of the Desargues' Theorem that we just proved, it follows that the triangles must be perspective from a line, i.e. the points $PC \cap MB = A$, $CZ \cap BY = O$ and $ZP \cap YM = X$ are collinear. With this, we proved that AX passes through O, so $\triangle ABC$ and $\triangle XYZ$ are perspective from a point. ■

Let's see it in action. We will give an alternate proof to Example 15.1.1:

Example 15.5.1 (Orthic axis)**.** Let AA_1, BB_1 and CC_1 be the altitudes in $\triangle ABC$. Let A' be the intersection of the lines BC and B_1C_1 and similarly define the points B' and C'. Prove that A', B' and C' lie on a line.



Proof. Since AA_1, BB_1 and CC_1 are concurrent at the orthocenter of $\triangle ABC$, the triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ are perspective from a point. Then, by the Deargues' Theorem, they are also perspective from a line, i.e. the points $AB \cap A_1B_1 = C'$, $BC \cap B_1C_1 = A'$ and $CA \cap C_1A_1 = B'$ are collinear. ■

Remark. This proof can be used for more generalized problem, where AA_1, BB_1 and CC_1 are any cevians in $\triangle ABC$ that are concurrent.

We will end this chapter here, but we must mention that collinearity plays an important role in the chapter Homothety, so we will continue this theme later in our journey.

Related problems: (Collinearity) 28, 97, 112 and 210.

Chapter 16

Concurrence II

16.1 Radical Center

Recall section 14.2, where we saw that the pairwise radical axes of three circles concur at the radical center. This is another approach of proving concurrence in geometry problems.

Example 16.1.1 (IMO 1995/1)**.** Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN and XY are concurrent.

Proof. We will prove that these three lines are radical axis of three circles. Let the circle with diameter AC be ω_1 and the circle with diameter BD be ω_2.

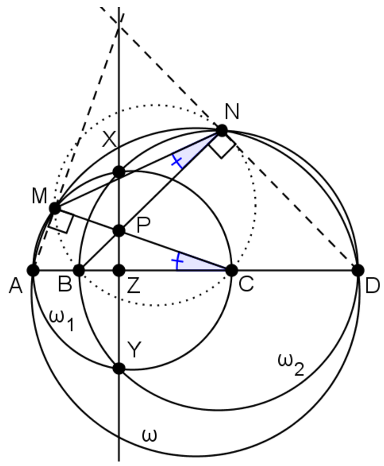
$$\overline{PM}\cdot\overline{PC} \overset{\omega_1}{=} \overline{PX}\cdot\overline{PY} \overset{\omega_2}{=} \overline{PB}\cdot\overline{PN}$$

$$\therefore BCNM \text{ is cyclic} \qquad (*)$$

Since AC and BD are diameters of ω_1 and ω_2, then $\angle AMC = 90^\circ = \angle BND$.

$$\begin{aligned}\angle MND &= \angle MNB + \angle BND \overset{(*)}{=}\\ &= \angle MCB + 90^\circ \equiv\\ &\equiv \angle MCA + 90^\circ \overset{\triangle AMC}{=}\\ &= 90^\circ - \angle MAC + 90^\circ \equiv\\ &\equiv 180^\circ - \angle MAD\end{aligned}$$

$$\therefore MADN \text{ is cyclic}$$

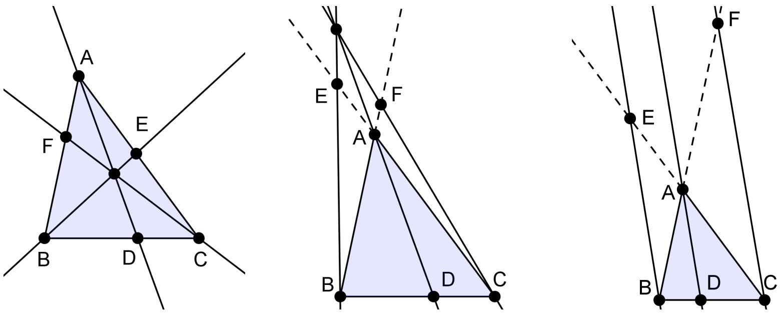


Now, we have three circles: $(MAYCX)$, $(MADN)$ and $(NXBYD)$. Their pairwise radical axes are MA, DN and XY, so they are concurrent at the radical center of these three circles. ■

16.2 Ceva's Theorem

Property 16.2.1 (Ceva's Theorem)**.** Let ABC be a triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, such that even number of them (zero or two) are on the extensions of the sides. The lines AD, BE and CF are concurrent or parallel if and only if

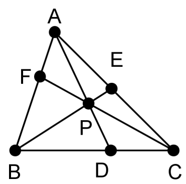
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



Remark. We write the ratio in exactly the same manner as we showed in Menelaus' Theorem.

Proof. Let the lines AD, BE and CF be concurrent at P. Assume that the point P is inside the triangle ABC. (When P is outside, the proof is similar)

$$\frac{P_{\triangle CAF}}{P_{\triangle CFB}} = \frac{\overline{AF}}{\overline{FB}}$$
$$\frac{P_{\triangle PAF}}{P_{\triangle PFB}} = \frac{\overline{AF}}{\overline{FB}}$$



$$\therefore \frac{P_{\triangle CAF} - P_{\triangle PAF}}{P_{\triangle CFB} - P_{\triangle PFB}} = \frac{\overline{AF}}{\overline{FB}}$$

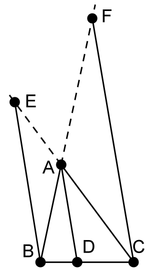
$$\text{i.e. } \frac{P_{\triangle CAP}}{P_{\triangle BCP}} = \frac{\overline{AF}}{\overline{FB}}.$$

$$\text{Similarly, } \frac{P_{\triangle ABP}}{P_{\triangle CAP}} = \frac{\overline{BD}}{\overline{DC}} \text{ and } \frac{P_{\triangle BCP}}{P_{\triangle ABP}} = \frac{\overline{CE}}{\overline{EA}}.$$

$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1. \quad \square$$

Let the lines AD, BE and CF be parallel. Exactly one of the points must be on the side of the triangle, WLOG let that point be D (the other two points are on the extensions of the sides). By Thales' Proportionality Theorem, we get

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{DC}}{\overline{CB}} \quad (\because DA \parallel CF)$$
$$\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CB}}{\overline{BD}} \quad (\because DA \parallel BE)$$
$$\therefore \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{DC}}{\overline{CB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CB}}{\overline{BD}} = 1. \quad \square$$

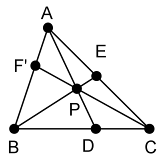


Now, let's prove the other direction. Let

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

and let even number of the points be on the extensions of the sides. Let the intersection of the lines AD and BE be P. In the case when there is no intersection, i.e. when $AD \parallel BE$, it can be easily proven that AD, BE and CF are parallel.

Let CP intersect AB at F'. Similarly as in the proof of Menelaus' Theorem, we are using the direction of Ceva's Theorem that we just proved (for AD, BE and CF' which do concur) and we get:



$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Combining with the condition, we get:

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AF'}}{\overline{F'B}}.$$

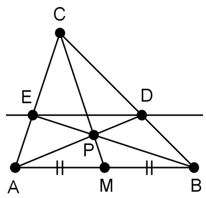
Keeping in mind that $\overline{AF} + \overline{FB} = \overline{AB} = \overline{AF'} + \overline{F'B}$, by adding 1 to both sides, we get:

$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AB}}{\overline{F'B}},$$

which means $\overline{FB} = \overline{F'B}$, i.e. $F \equiv F'$. We should note that in the last part, we assumed that F' is between A and B. That is a safe assumption because there are either zero or two points on the extensions of the sides; If there are zero, then D and E are on the sides BC and CA, so F' must also lie on the side AB; If there are two points on the extensions, then WLOG let them be D and E and F' will again lie on the side AB. ■

In the next few examples, we will show how we can use Ceva's Theorem in both directions.

Example 16.2.1. In $\triangle ABC$, let M be the midpoint of the side AB. Let P be an arbitrary point on the segment CM ($P \neq C$, $P \neq M$). Let $AP \cap BC = D$ and $BP \cap AC = E$. Prove that $ED \parallel AB$.



Proof. The lines AD, BE and CM are concurrent, so we can use Ceva's Theorem:

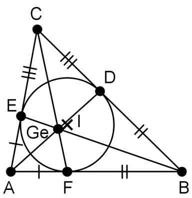
$$\frac{\overline{AM}}{\overline{MB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Since M is the midpoint of AB, $\overline{AM} = \overline{MB}$, so by canceling and then rearranging, we get:

$$\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CD}}{\overline{DB}},$$

which by Thales' Proportionality Theorem means that $ED \parallel AB$. ■

Property 16.2.2 (Gergonne Point)**.** Let D, E and F be the tangent points of the incircle of $\triangle ABC$ with the sides BC, CA and AB, respectively. Prove that AD, BE and CF are concurrent.



Proof. $\overline{AF} = \overline{AE} = x$ as tangent segments from the point A to the incircle. Similarly, $\overline{BF} = \overline{BD} = y$ and $\overline{CD} = \overline{CE} = z$.

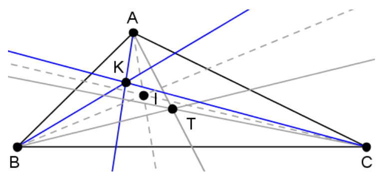
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} = 1,$$

so by Ceva's Theorem, AD, BE and CF are concurrent. ■

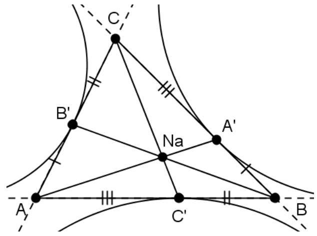
Remark. This point of concurrence is known as the *Gergonne Point* of the triangle ABC.

Property 16.2.3 (Lemoine Point)**.** The three symmedians in a triangle are concurrent.

Proof. Using Property 13.1, by Ceva's Theorem, it immediately follows that the three symmedians in a triangle are concurrent. This point of concurrence is called the *Lemoine Point* of the triangle. ■



Property 16.2.4 (Nagel Point)**.** Let A', B' and C' be the tangent points of the A-excircle, B-excircle and C-excircle with the sides BC, CA and AB in the $\triangle ABC$, respectively. Prove that AA', BB' and CC' are concurrent.



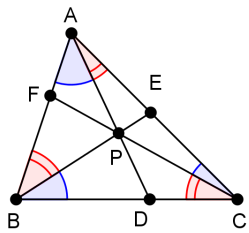
Proof. From Property 10.1.3, we know that $\overline{AB} + \overline{BA'} = \overline{AC} + \overline{CA'}$. We see that LHS and RHS add up to the perimeter of $\triangle ABC$, so each of them is equal to the semiperimeter s. Therefore, $\overline{BA'} = s - c$ and $\overline{CA'} = s - b$. Similarly, $\overline{CB'} = s - a$, $\overline{AB'} = s - c$, $\overline{AC'} = s - b$ and $\overline{BC'} = s - a$.

$$\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = \frac{s-b}{s-a} \cdot \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} = 1,$$

so by Ceva's Theorem, AA', BB' and CC' are concurrent at the *Nagel Point*. ■

Property 16.2.5 (Trigonometric Ceva's Theorem)**.** Given a triangle ABC and points D, E and F that lie on the lines BC, CA and AB, respectively; the lines AD, BE and CF are concurrent or parallel if and only if

$$\frac{\sin\angle BAD}{\sin\angle CAD} \cdot \frac{\sin\angle CBE}{\sin\angle ABE} \cdot \frac{\sin\angle ACF}{\sin\angle BCF} = 1.$$



Proof. By using the Law of Sines in $\triangle ABD$, we get

$$\frac{\overline{BD}}{\sin\angle BAD} = \frac{\overline{AB}}{\sin\angle BDA}, \text{ i.e. } \sin\angle BAD = \frac{\overline{BD} \cdot \sin\angle BDA}{\overline{AB}}.$$

Similarly, for $\triangle ACD$, we get $\sin\angle CAD = \dfrac{\overline{CD} \cdot \sin\angle CDA}{\overline{AC}}$.

Since $D \in BC$, then the angles $\angle BDA$ and $\angle CDA$ are always equal or supplementary. Therefore, $\sin\angle BDA = \sin\angle CDA$. By dividing the previous equations, we get

$$\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{\overline{BD}}{\overline{CD}} \cdot \frac{\overline{AC}}{\overline{AB}}.$$

Analogously, we can get similar equations for the cevians BE and CF. By multiplying these three equations, we get

$$\frac{\sin\angle BAD}{\sin\angle CAD} \cdot \frac{\sin\angle CBE}{\sin\angle ABE} \cdot \frac{\sin\angle ACF}{\sin\angle BCF} = \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}},$$

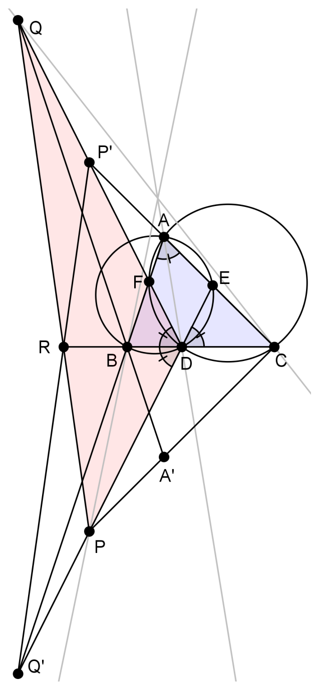
so by Ceva's Theorem we are done. ■

16.3 Desargues' Theorem

Here is an example that shows how we can use Desargues' Theorem when we need to prove concurrence.

Example 16.3.1 (RMM 2016)**.** Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle (ABD) meets the segment AC again at an interior point E. The circle (ACD) meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).

Proof. Let σ denote reflection in the line BC. Since $\angle BDF = \angle BAC = \angle CDE$ (because of the cyclic quadrilaterals $ABDE$ and $ACDF$), the lines DE and DF are images of one another under σ, so the lines AC and DF meet at $P' \equiv \sigma(P)$, and the lines AB and DE meet at $Q' \equiv \sigma(Q)$. Consequently, the lines PQ and $P'Q' \equiv \sigma(PQ)$ meet at some point R on the line BC. Since the points $Q' = AB \cap DP$, $R = BC \cap PQ$ and $P' = CA \cap QD$ are collinear, the triangles $\triangle ABC$ and $\triangle DPQ$ are perspective from a a line. Therefore, by Desargues' Theorem, they are also perspective from a point, i.e. the lines AD, BP and CQ are concurrent. ■



Related problems: (Concurrence) 101, 191 and 197. (Ceva) 179.

Chapter 17

Homothety

Definition and properties

A homothety with center O and ratio k is a function that sends every point on the plane P to a point P' such that

$$\overrightarrow{OP'} = k \cdot \overrightarrow{OP}.$$

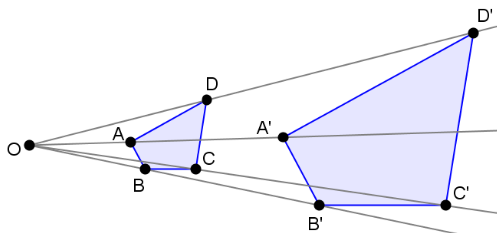


Figure 17.1: A homothety with center O and ratio $k = 2.5$

From this definition, we can directly conclude the following properties:

Property 17.1. The image point, the original point and the center of the homothety are collinear.

Property 17.2. A homothety always sends a figure to a similar figure, such that the corresponding sides are parallel.

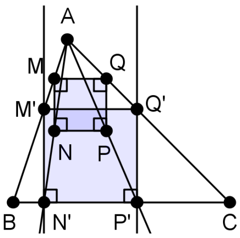
If $k > 0$, then the image and the original will be on the same side of the center; If $k < 0$, the image and the original will be on different sides of the center, i.e. the center will be between them. If $|k| > 1$, then the homothety is a magnification; If $|k| < 1$, then it is a reduction.

We will use the notation $\mathcal{X}_{O,\,k} : P \to P'$ to denote that P' is the image of P under the homothety centered at O with ratio k.

Getting started

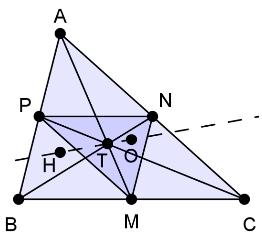
As an exercise, let's try to construct a square that is "inscribed" in a $\triangle ABC$, such that one vertex lies on the side AB, one on the side AC and two adjacent vertices of the square lie on the side BC. Firstly try by yourself and then see the solution presented below.

Construct any square $MNPQ$ such that $M \in AB$ and $Q \in AC$ and $MQ \parallel NP \parallel BC$. We will now define a homothety centered at A that will send $MNPQ$ to the desired square. Let $AN \cap BC = N'$. We define the ratio of the homothety $k = \dfrac{\overline{AN'}}{\overline{AN}}$, so that $\mathcal{X} : N \to N'$. Since $NP \parallel BC$ and $N' \in BC$, the image of P will be a point P' on BC. Also the center of the homothety (A), the original (P) and the image (P') must be collinear, so $P' = AP \cap BC$.



Now, let's find M'. $M'N'$ should be parallel to MN, but also $A - M - M'$ should be collinear, so M' is the intersection of the perpendicular to BC through N' and the line $AM \equiv AB$. We can find Q' similarly to M'. The resulting quadrilateral $M'N'P'Q'$ is similar to its original $MNPQ$, so it must be a square. It is also "inscribed" in $\triangle ABC$ per the given conditions, so we are done.

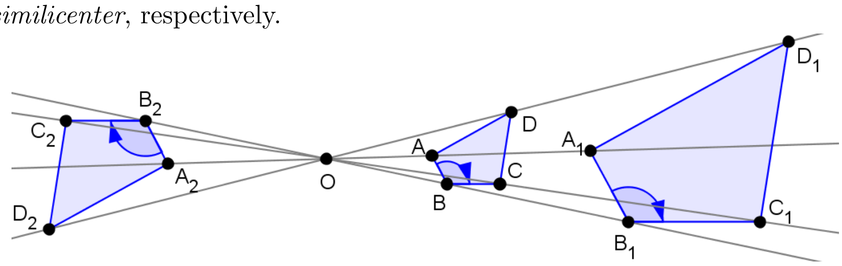
As another exercise, we will give an alternate proof of Property 10.3.3, where we proved that in any triangle ABC, the orthocenter H, the centroid T and the circumcenter O are collinear and that $\overline{HT} = 2 \cdot \overline{TO}$.



Let M, N and P be the midpoints of the sides BC, CA and AB, respectively. Remember that the medians AM, BN and CP intersect at the centroid T and moreover, it divides them in ratio 2:1. Therefore, $\mathcal{X}_{T,-\frac{1}{2}} : ABC \to MNP$. Also, $\mathcal{X} : H_{ABC} \to H_{MNP}$, so the points $H_{ABC} - T_{ABC} - H_{MNP}$ are collinear. From Property 6.10, we know that $O_{ABC} \equiv H_{MNP}$, so $H_{ABC} - T_{ABC} - O_{ABC}$ are collinear. Because $|k| = \frac{1}{2}$, we can also conclude that $\overline{HT} = 2 \cdot \overline{TO}$.

17.1 Homothetic Center of Circles

Homothetic centers may be external ($k > 0$) or internal ($k < 0$). If the center is internal, the two geometric figures are scaled, 180°-rotated and translated images of one another. Otherwise, if the center is external, the two figures are scaled and translated similar to one another. Sometimes, the external and internal homothetic centers (centers of similitude) are called *exsimilicenter* and *insimilicenter*, respectively.



Circles are geometrically similar to one another and "rotation invariant". Hence, a pair of circles has both types of homothetic centers, internal and external (unless the centers coincide or the radii are equal; we will discuss these special cases later). These two homothetic centers lie on the line joining the centers of the two given circles.

How can we find those homothetic centers? Let's draw two parallel diameters A_1B_1 and A_2B_2, one for each circle. These make the same angle with the line connecting the centers. The lines A_1A_2, and B_1B_2, intersect each other and the line connecting the centers at the external homothetic center. Conversely, the lines A_1B_2 and B_1A_2 intersects each other and the line connecting the centers at the internal homothetic center. As a limiting case of this construction, a line tangent to both circles passes through one of the homothetic centers, as it forms right angles with both the corresponding diameters, which are thus parallel. The common external tangents pass through the external homothetic center, while the common internal tangents pass through the internal homothetic center. If the circles have the same radius (but different centers), they have no external homothetic center. If the circles have the same center, they have only one homothetic center and that is the common center of the circles.

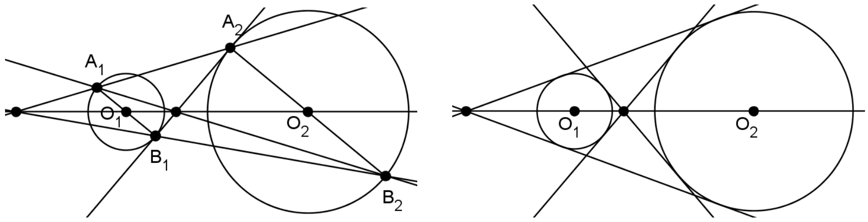
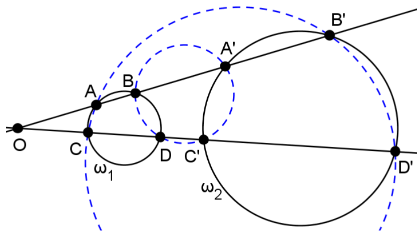


Figure 17.2: Internal and external homothetic center of two circles.

A line through a homothetic center that intersects the circles, will intersect each circle at two places. Of these four points, any two are said to be *homologous* if radii drawn to them make the same angle with the line connecting the centers (eg. A and A'). Out of these four, any two that lie on different circles and are not homologous are said to be *antihomologous* (eg. A and B').



We will now prove that any two pairs of antihomologous points (defined by lines through the same homothetic center) are concyclic. Let O be a homothetic center of ω_1 and ω_2. Let a line through O intersect ω_1 at A and B and ω_2 at A' and B' (such that A and A' are closer to O than B and B', respectively). Then,

$$\overline{OA} \cdot \overline{OB'} = \overline{OA} \cdot (k \cdot \overline{OB}) = k \cdot \overline{OA} \cdot \overline{OB}.$$

If we similarly define points C, D, C' and D' for a different line through O, we have

$$\overline{OC} \cdot \overline{OD'} = \overline{OC} \cdot (k \cdot \overline{OD}) = k \cdot \overline{OC} \cdot \overline{OD}.$$

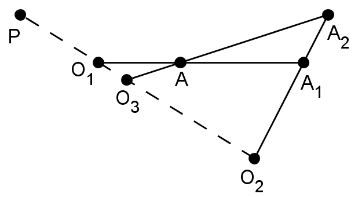
From the intersecting secants theorem for ω_1, we have $\overline{OA} \cdot \overline{OB} = \overline{OC} \cdot \overline{OD}$, so combining the previous equations, we get $\overline{OA} \cdot \overline{OB'} = \overline{OC} \cdot \overline{OD'}$ which means that the points A, B', C and D' (which are two pairs of antihomologous points) are concyclic. Just as a trivia for our more curious readers, we will mention that because of this concyclicity, the intersection of the lines AC and $B'D'$ lies on the radical axis of the circles ω_1 and ω_2. Can you see why?

17.2 Composition of Homotheties

Composition of two homotheties, $\mathcal{X}_1$ with center O_1 and ratio k_1, and $\mathcal{X}_2$ with center O_2 and ratio k_2 is a homothety $\mathcal{X}_3$ (unless $k_1 \cdot k_2 = 1$).

Property 17.3. Let $\mathcal{X}_3$ be the composition of homotheties $\mathcal{X}_2 \circ \mathcal{X}_1$. Then, the center of $\mathcal{X}_3$ lies on the line O_1O_2 and the ratio of $\mathcal{X}_3$ is $k_1 \cdot k_2$.

Proof. Let A be a point. Let A_1 and A_2 be points such that $\mathcal{X}_1 : A \to A_1$ and $\mathcal{X}_2 : A_1 \to A_2$. Since $\mathcal{X}_3(A) = \mathcal{X}_2 \circ \mathcal{X}_1(A) = \mathcal{X}_2(\mathcal{X}_1(A)) = \mathcal{X}_2(A_1) = A_2$, we get $\mathcal{X}_3 : A \to A_2$.



Let's prove that the center of $\mathcal{X}_3$, O_3, lies on O_1O_2. Let $\mathcal{X}_2 : O_1 \to P$. The point P doesn't have a special meaning, but it will help us prove our claim. We will also use the fact that the center of homothety is fixed under a homothety, i.e. $\mathcal{X}_1 : O_1 \to O_1$.

$$\mathcal{X}_3(O_1) = \mathcal{X}_2(\mathcal{X}_1(O_1)) = \mathcal{X}_2(O_1) = P$$

From this equation, we have $\mathcal{X}_3 : O_1 \to P$ and $\mathcal{X}_2 : O_1 \to P$. Using Property 17.1, which says that the center of homothety, the original, and the image are collinear, we have $O_3 - O_1 - P$ and $O_2 - O_1 - P$, i.e. all four points are collinear, so the center of $\mathcal{X}_3$, which is O_3, lies on O_1O_2.

We will now find the ratio of $\mathcal{X}_3$. Let's get back to the points A, A_1 and A_2 that we defined earlier. By definition of A_1, we get $\overline{O_1A_1} = k_1 \cdot \overline{O_1A}$. By definition of A_2, we get $\overline{O_2A_2} = k_2 \cdot \overline{O_2A_1}$. Because $\mathcal{X}_3 : A \to A_2$, we have $\overline{O_3A_2} = k_3 \cdot \overline{O_3A}$. Keep in mind the fact that we just proved, that O_3 is collinear with O_1 and O_2. Because of this collinearity, we can apply Menelaus' Theorem to $\triangle AA_1A_2$ and the points $O_2 - O_3 - O_1$

$$\frac{\overline{AO_1}}{\overline{O_1A_1}} \cdot \frac{\overline{A_1O_2}}{\overline{O_2A_2}} \cdot \frac{\overline{A_2O_3}}{\overline{O_3A}} = 1$$

$$\frac{1}{k_1} \cdot \frac{1}{k_2} \cdot \frac{k_3}{1} = 1$$

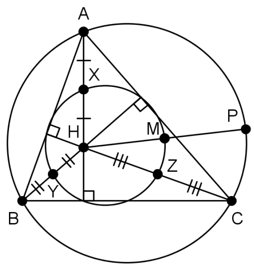
Finally, $k_3 = k_1 \cdot k_2$. ∎

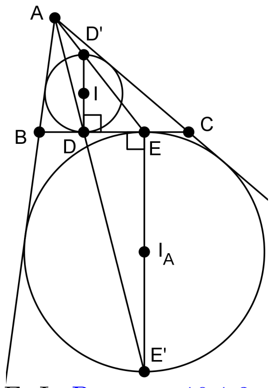
Related problems: (Homothety) 157, 187, 196, 214 and 229.

17.3 Useful Lemmas

Property 17.4. Prove that the nine point circle bisects any line segment from the orthocenter to the circumcircle.

Proof. Let ABC be a triangle with orthocenter H. Let X, Y and Z be the midpoints of AH, BH and CH, respectively. Then, there is a homothety centered at H, with ratio 2, that sends $\triangle XYZ$ to $\triangle ABC$. Since the circumcirle of $\triangle XYZ$ is the nine point circle of $\triangle ABC$, this homothety sends the nine point circle of $\triangle ABC$ to its circumcirle. Let P be any point on the circumcircle of $\triangle ABC$. Let the nine point circle intersect HP at M. Then, $\mathcal{X}_{H,2} : M \to P$. Therefore, $\overline{HP} = 2 \cdot \overline{HM}$, i.e. $\overline{HM} = \overline{MP}$. ■





Property 17.5 (Diameter of the incircle)**.** Let the incircle of $\triangle ABC$ touch the side BC at D and let DD' be a diameter of the incircle. Let $AD' \cap BC = E$. Prove that $\overline{BD} = \overline{EC}$.

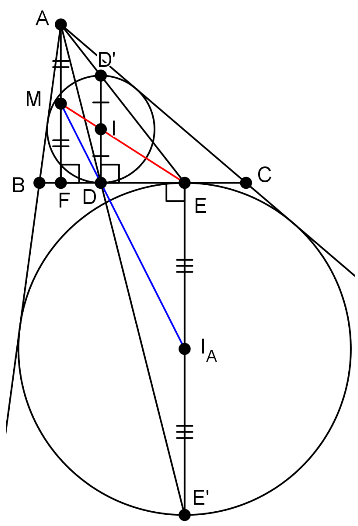
Proof. Consider the homothety with center A that sends the incircle to the A-excircle. The diameter DD' of the incircle must be mapped to the diameter of the excircle that is perpendicular to BC. It follows that D' must get mapped to the point of tangency between the excircle and BC. Since the image of D' must lie on the line AD', it must be E. That is, the excircle is tangent to BC at E. In Property 10.1.3, we already proved that the tangent points of the incircle and the excircle to BC are equidistant from the midpoint, so $\overline{BD} = \overline{EC}$. ■

Remark (1). Similarly, because of the same homothety, if EE' is diameter of the excircle, then $A - D - E'$ are collinear.

Remark (2). Notice that, as a consequence to these collinearities, the line joining the incenter and the midpoint of BC is parallel to the line AE, while the line joining the $A-$excenter and the midpoint of BC is parallel to the line AD.

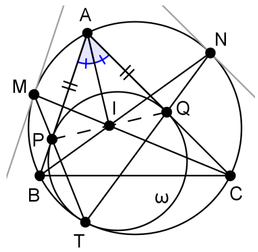
Property 17.6 (Midpoint of the altitude)**.** Let ABC be a triangle and let D and E be the tangent points of the side BC with the incircle (centered at I) and $A-$excircle (centered at I_A), respectively. If M is the midpoint of the altitude AF, prove that $M = EI \cap DI_A$.

Proof. From Property 17.5, we know that $A - D' - E$ are collinear. Since $AF \parallel D'D$, the homothety centered at E that takes DD' to FA also takes the midpoint I of DD' to the midpoint M of FA and therefore $E - I - M$ are collinear.
Similarly, $A - D - E'$ are collinear, so the homothety centered at D (with negative coefficient) that takes EE' to FA also takes I_A to M and therefore $I_A - D - M$ are collinear. ■



Property 17.7. Let ABC be a triangle. A circle ω is internally tangent to the circumcircle of $\triangle ABC$ and also to the sides AB and AC at P and Q, respectively. Prove that the midpoint of PQ is the incenter of $\triangle ABC$.

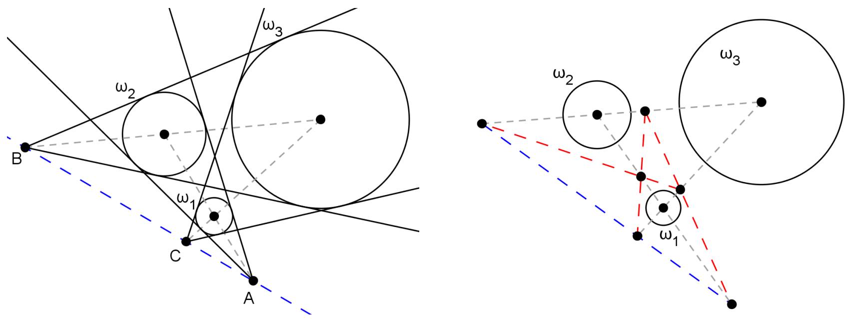
Proof. Let the tangent point of (ABC) and ω be T. Let $\mathcal{X}$ be a homothety that sends ω to (ABC), i.e. $\mathcal{X}_T : \omega \to (ABC)$. Let $TP \cap (ABC) = M$. Since AB is tangent to ω at P, the parallel line to AB through M should be tangent to (ABC) at M. This means that M must be the midpoint of the arc $\overset{\frown}{AB}$ in (ABC). So CM is an angle bisector in $\triangle ABC$, i.e. $C - I - M$ are collinear, where I is the incenter in $\triangle ABC$. Let $TQ \cap (ABC) = N$. Similarly, $B - I - N$ are collinear. By applying Pascal's Theorem to the points T, M, C, A, B and N, we get that the points $P - I - Q$ are collinear. Also, $\overline{AP} = \overline{AQ}$ as tangent segment, and AI is the angle bisector of $\angle BAC \equiv \angle PAQ$, so I must be the midpoint of PQ. ■



Property 17.8 (Monge's Theorem)**.** The exsimilicenters of three circles are collinear.

Proof. Let ω_1, ω_2 and ω_3 be three circles. Let A, B and C be the intersections of the external tangents of ω_1 and ω_2; ω_2 and ω_3; and ω_1 and ω_3, respectively. One of the two homotheties that sends ω_1 to ω_2 is centered at A and has a coefficient $k_1 > 0$, i.e. $\mathcal{X}_{A,\,k_1} : \omega_1 \to \omega_2$. Similarly, $\mathcal{X}_{B,\,k_2} : \omega_2 \to \omega_3$, where $k_2 > 0$. Therefore, the composition homothety $\mathcal{X}_{comp} = \mathcal{X}_B \circ \mathcal{X}_A$ sends ω_1 to ω_3. By the properties of composition of homotheties, we know that the center of $\mathcal{X}_{comp}$ lies on the line AB and the coefficient is positive (as it is equal to $k_1 \cdot k_2$). But the center of the homothety that sends ω_1 to ω_3 with positive coefficient is found as the intersection of the common external tangents, so it is C. In conclusion, $C \in AB$, i.e. the points A, B and C are collinear. ■

Remark. Can you prove this using Menelaus' Theorem? What about using Desargues' Theorem?



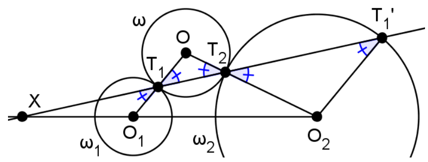
(a) Monge's Theorem

(b) Monge-d'Alembert Theorem

Property 17.9 (Monge-d'Alembert Theorem)**.** Given three circles, the insimilicenters of any two pairs of circles and the exsimilicenter of the third one are collinear.

Proof. The proof is analogous to the proof of Monge's Theorem. ■

Property 17.10. Let ω be a circle that is tangent to ω_1 and ω_2 at T_1 and T_2, respectively. Prove that the line T_1T_2 passes through one of the homothetic centers of ω_1 and ω_2.



Proof 1. We will discuss the case where ω is externally tangent to both ω_1 and ω_2. The other cases should be analogous. Let O, O_1 and O_2 be the centers of ω, ω_1 and ω_2, respectively.

$$\angle OT_1T_2 = \angle OT_2T_1 \quad (\because \overline{OT_1} = \overline{OT_2})$$

Let $X = T_1T_2 \cap O_1O_2$. Because of the tangency of ω and ω_1, we know that $T_1 \in OO_1$, so

$$\angle OT_1T_2 = \angle O_1T_1X.$$

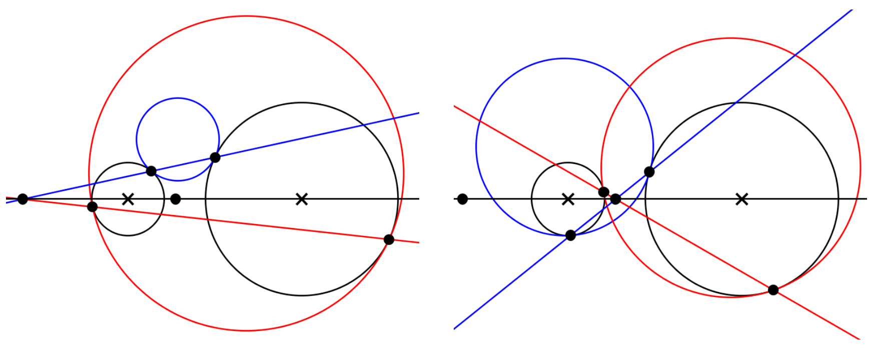
Let $T_1' \in T_1T_2 \cap \omega_2$. From the tangency of ω and ω_2, we know that $T_2 \in OO_2$, so

$$\angle OT_2T_1 = \angle O_2T_2T_1' = \angle O_2T_1'T_2 \equiv \angle O_2T_1'X.$$

Combining the three equations, we get $\angle O_1T_1X = \angle O_2T_1'X$, so X is a homothetic center of ω_1 and ω_2. ∎

Proof 2. Using the same case and the same notations as in the previous proof, the tangent point T_1 is the insimilicenter of the tangent circles ω and ω_1. Similarly, T_2 is the insimilicenter of ω and ω_2. Therefore, by Monge-d'Alembert Theorem, T_1T_2 passes through the exsimilicenter of ω_1 and ω_2. ∎

Remark. The line T_1T_2 passes through the external homothetic center of ω_1 and ω_2 when ω is either internally or externally tangent to both ω_1 and ω_2. Otherwise, when ω is internally tangent to one of ω_1 and ω_2 and externally tangent to the other one, then the line T_1T_2 passes through the internal homothetic center of ω_1 and ω_2.



Related problems: (Lemmas) 181, 207, 212, 216, 220, 222 and 226.

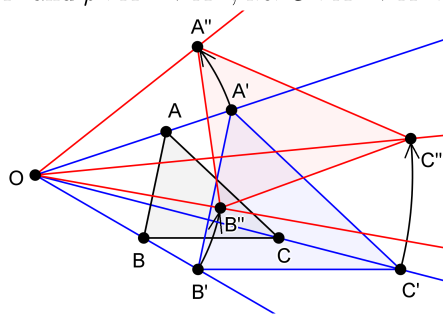
Chapter 18

Spiral Similarity

A *spiral similarity* is a function that sends one point in the plane to another. It is a composition of homothety and rotation with the same center. More formally, if we denote the spiral similarity by $\mathcal{S}$, the rotation centered at O with angle φ by $\rho_{O,\,\varphi}$, and the homothety centered at O with ratio k by $\mathcal{X}_{O,\,k}$, then

$$\mathcal{S}_{O,\,k,\,\varphi} = \rho_{O,\,\varphi} \circ \mathcal{X}_{O,\,k}$$

Let's make this more clear by showing an example. Let ABC be a triangle and let O be a point in its plane. We will show how to construct the image of $\triangle ABC$ with respect to the spiral similarity $\mathcal{S}$ centered at O with ratio $k = 1.5$ and angle $\varphi = 20°$. We will consider one vertex at a time and then connect the images of the vertices to get the image of the triangle. Firstly, we will find the images of the homothety centered at O with $k = 1.5$ and later we will rotate those images with center O and $\varphi = 20°$ in the positive (counter-clockwise) direction in order to find the images of the spiral similarity. We will use the notations $\mathcal{X} : X \to X'$ and $\rho : X' \to X''$, i.e. $\mathcal{S} : X \to X''$.



Since the image of a triangle after a homothety is a triangle similar to the original and the image of a triangle after a rotation is a triangle congruent to the original, we get that $\triangle ABC \sim \triangle A''B''C''$ (with a ratio of similarity $|k|$). Also, since

$$\frac{\overline{OA''}}{\overline{OA}} = \frac{\overline{OA'}}{\overline{OA}} = |k| = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{OB''}}{\overline{OB}} \quad \text{and}$$

$$\angle A''OB'' = \angle A''OA + \angle AOB'' = \varphi + \angle AOB - \angle BOB'' = \varphi + \angle AOB - \varphi = \angle AOB,$$

we get that $\triangle OA''B'' \sim \triangle OAB$ (with a ratio of similarity $|k|$).

Notice that since $AB \parallel A'B'$ we have $\measuredangle(AB, A''B'') = \measuredangle(A'B', A''B'') = \varphi$.

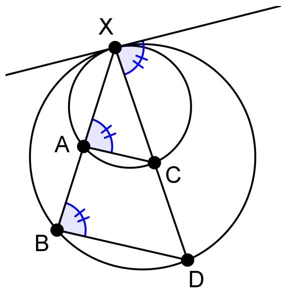
Now let's say we have two line segments AB and CD and we want to find if there is a spiral similarity that sends A to C and B to D, i.e. $\mathcal{S} : AB \to CD$. Since a spiral similarity is defined by a center O, a ratio k and an angle φ, we need to find those 3 values.

We can find the ratio k and the angle φ easily because, as previously mentioned, $k = \frac{\overline{CD}}{\overline{AB}}$ and $\varphi = \measuredangle(AB, CD)$. We now have two cases:

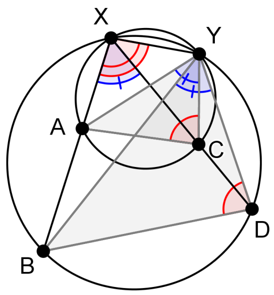
i) $AB \nparallel CD$.

Let $AB \cap CD = X$.
Then, since $\mathcal{S} : A \to C$, $\measuredangle(AO, OC) = \varphi = \measuredangle(AB, CD) \equiv \measuredangle(AX, XC)$ which means that A, O, C and X are concyclic[1]. Similarly, since $\mathcal{S} : B \to D$, we get that B, O, D and X are concyclic. This means that the center of spiral similarity lies on (XAC) and (XBD). We now have two sub-cases:

(a) $AC \parallel BD$.
Then, $\angle XAC = \angle XBD$. Therefore, since $XC \equiv XD$, by Property 5.3 we get that the circles (XAC) and (XBD) have a common tangent at X, i.e. they only have one common point. So, X is the only candidate for the center of spiral similarity. We only have to check if it satisfies the ratio condition. Since $AC \parallel BD$, we get that $\triangle XAC \sim \triangle XBD$ and therefore $\dfrac{\overline{XA}}{\overline{XC}} = \dfrac{\overline{XB}}{\overline{XD}}$. Thus, $X \equiv O$.



(a)



(b)

(b) $AC \nparallel BD$.
Then, $\angle XAC \neq \angle XBD$ and similarly as in the previous case, we get that the circles (XAC) and (XBD) are not tangent to each other, which means that they have another intersection $Y \neq X$. Since $AC \nparallel BD$, then $\dfrac{\overline{XA}}{\overline{XC}} \neq \dfrac{\overline{XB}}{\overline{XD}}$, which means that X can not be the center of spiral similarity. Then, Y is the only other candidate. We have to check if it satisfies the ratio condition and the angle condition.
Since Y lies on the circles (XAC) and (XBD), we have $\measuredangle(YA, YC) = \measuredangle(XA, XC) \equiv \measuredangle(XB, XD) = \measuredangle(YB, YD)$ and $\measuredangle(CA, CY) = \measuredangle(XA, XY) \equiv \measuredangle(XB, XY) = \measuredangle(DB, DY)$.
From here, we get that $\triangle YAC \sim \triangle YBD$ from where we get that both conditions are satisfied. We conclude that $Y \equiv O$.

[1] Here we use the notation $\measuredangle(AO, OC)$ for the *directed* angle between the lines AO and OC, always in the same direction (for example, in the positive direction). Thus, even if O and X are on different sides of the line AC, these four points will still be concyclic.

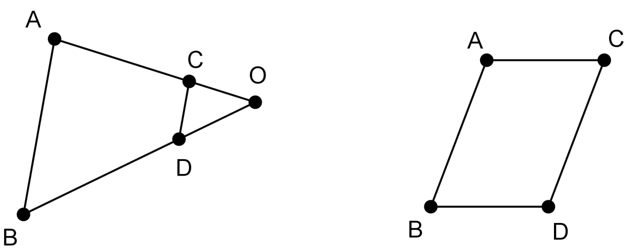
ii) $AB \parallel CD$.

Then, $\varphi = \measuredangle(AB, CD) = 0°$. This means that there is no rotation after the homothety, so the center of the spiral similarity is the center of the homothety that sends AB to CD and we can find it easily because (from Property 17.1) we know that the center of homothety, the original and the image are collinear, so $O - A - C$ and $O - B - D$ should be collinear. However, this is the only case where k can be negative, so the segments in the ratio should be considered as directed segments (meaning the ratio is positive if they have the same direction, but negative if they have opposite directions).

We now have two sub-cases:

(a) $AC \nparallel BD$.
Then, $O = AC \cap BD$ and we are done.

(b) $AC \parallel BD$.
The lines AC and BD do not intersect, so there is no homothety that sends AB to CD and consequently, there does not exist a spiral similarity that sends AB to CD in this case (when $ABDC$ is a parallelogram).

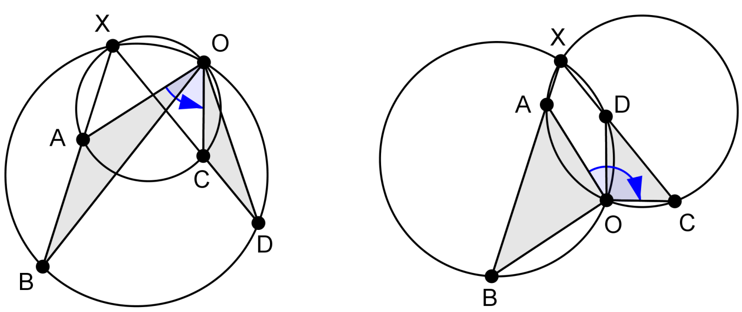


(a) (b)

From the above discussion, we can conclude the following properties.

Property 18.1 (Uniqueness)**.** If a spiral similarity $\mathcal{S} : AB \to CD$ exists, then it must be unique.

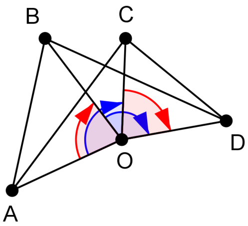
Property 18.2 (Find the center 1)**.** Let AB and CD be two segments such that $AB \nparallel CD$ and $AC \nparallel BD$. Let $AB \cap CD = X$ and $(XAC) \cap (XBD) = O \neq X$. Then, O is the center of the spiral similarity $\mathcal{S} : AB \to CD$.



We will now present and prove a few other useful properties.

Property 18.3 (Same center)**.** Let O be the center of the spiral similarity $\mathcal{S}_1 : AB \to CD$. Then O is also the center of the spiral similarity $\mathcal{S}_2 : AC \to BD$.

Proof. In order to prove that O is the center of $\mathcal{S}_2 : AC \to BD$, we have to show that $\angle AOB = \angle COD$ and $\dfrac{\overline{OB}}{\overline{OA}} = \dfrac{\overline{OD}}{\overline{OC}}$.



Since O is the center of $\mathcal{S}_1 : AB \to CD$ we know that $\angle AOC = \angle BOD$ and $\dfrac{\overline{OC}}{\overline{OA}} = \dfrac{\overline{OD}}{\overline{OB}}$. Therefore,

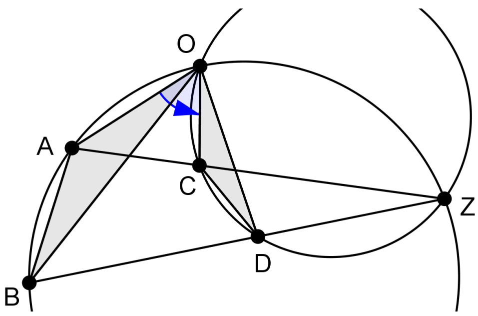
$$\begin{aligned}\measuredangle(AO, OB) &= \measuredangle(AO, OC) - \measuredangle(BO, OC) = \\ &= \measuredangle(BO, OD) - \measuredangle(BO, OC) = \measuredangle(CO, OD)\end{aligned}$$

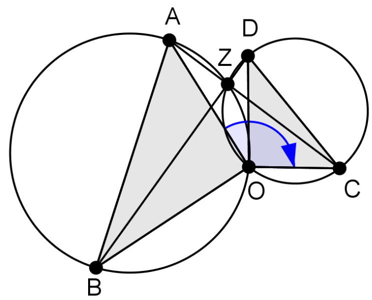
The ratio condition follows directly. ■

Remark. By combining the previous two properties, we can get yet another way of constructing the center of the spiral similarity $\mathcal{S} : AB \to CD$, presented in the following property.

Property 18.4 (Find the center 2)**.** Let AB and CD be two segments. Let $AC \cap BD = Z$ and let $(ZAB) \cap (ZCD) = O \neq Z$. Then, O is the center of the spiral similarity $\mathcal{S} : AB \to CD$.

Proof. Let $\mathcal{S}_1 : AC \to BD$. Then, by Property 18.2, O is the center of $\mathcal{S}_1$. Now, by Property 18.3, we get that O is also the center of $\mathcal{S}_2 : AB \to CD$. ■



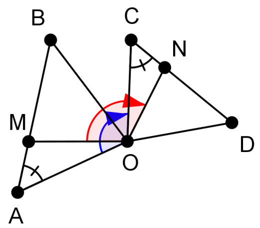


Property 18.5 (Corresponding point)**.** Let $\mathcal{S} : AB \to CD$ and let $M \in AB$, $N \in CD$ be points such that $\dfrac{\overline{AM}}{\overline{MB}} = \dfrac{\overline{CN}}{\overline{ND}}$. Then, $\mathcal{S} : M \to N$.

Proof. Let O be the center of $\mathcal{S}$. Then, $\triangle OAB \sim \triangle OCD$. From the condition, we can get $\dfrac{\overline{AM}}{\overline{AB}} = \dfrac{\overline{CN}}{\overline{CD}}$. From these two, we get

$$\frac{\overline{OA}}{\overline{OC}} = \frac{\overline{AB}}{\overline{CD}} = \frac{\overline{AM}}{\overline{CN}},$$

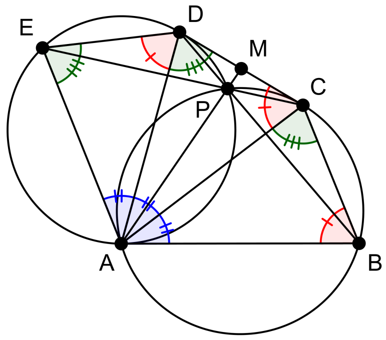
which when combined with $\angle OAM \equiv \angle OAB = \angle OCD \equiv \angle OCN$ implies that $\triangle OAM \sim \triangle OCN$. Therefore, $\mathcal{S} : AM \to CN$, i.e. $\mathcal{S} : M \to N$. ■



Remark. Keep in mind that the ratio condition may not always be given as such. It may be given that M and N are midpoints of AB and CD, respectively, from where the ratio clearly follows.

We will now solve a few problems using these properties.

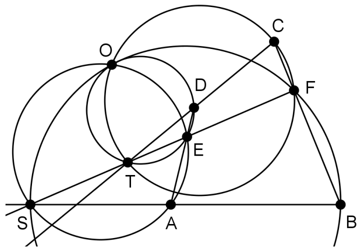
Example 18.1 (IMO Shortlist 2006)**.** Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle CBA = \angle DCA = \angle EDA$. Diagonals BD and CE meet at P. Prove that AP bisects side CD.



Proof. From the angle conditions, by the AA criterion, we get $\triangle ABC \sim \triangle ADE$. So A is the center of the spiral similarity that sends BC to DE. By Property 18.4, since P is the intersection of BD and CE we get that A lies on the circumcircles of $\triangle PBC$ and $\triangle PDE$, i.e. $ABCP$ and $APDE$ are cyclic. From $\angle ABC = \angle ACD$ we get that CD is tangent to the circumcircle of $\triangle ABC$. In addition, $\angle AED = \angle ADC$ so CD is also tangent to the circumcircle of $\triangle AED$. Finally if we let M be the intersection of AP and CD we can finish by Secant-Tangent Theorem.

$$\overline{MD}^2 = \overline{MP} \cdot \overline{MA} = \overline{MC}^2, \text{ i.e. } \overline{MD} = \overline{MC}$$ ■

Example 18.2 (USAMO 2006)**.** Let $ABCD$ be a quadrilateral and let E and F be points on sides AD and BC, respectively, such that $\dfrac{\overline{AE}}{\overline{ED}} = \dfrac{\overline{BF}}{\overline{FC}}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of $\triangle SAE$, $\triangle SBF$, $\triangle TCF$ and $\triangle TDE$ pass through a common point.



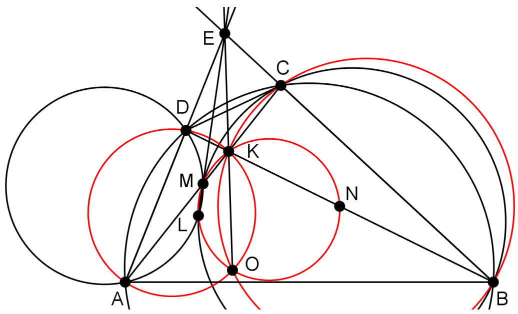
Proof. Let O be the center of the spiral similarity $\mathcal{S}$ that sends AD to BC. Then, $\mathcal{S} : A \to B$, $\mathcal{S} : D \to C$ and by Property 18.5 $\mathcal{S} : E \to F$.

Then, $\mathcal{S} : AE \to BF$. By Property 18.4, since $AB \cap EF = S$, we get $O \in (SAE)$ and $O \in (SBF)$.

Also, $\mathcal{S} : ED \to FC$. By Property 18.4, since $EF \cap DC = T$, we get $O \in (TED)$ and $O \in (TFC)$.

Therefore, these four circles pass through the center O of $\mathcal{S}$. ■

Example 18.3 (International Zhautykov Olympiad 2011)**.** Diagonals of a cyclic quadrilateral $ABCD$ intersect at point K. The midpoints of diagonals AC and BD are M and N, respectively. The circumcircles of $\triangle ADM$ and $\triangle BCM$ intersect at points M and L. Prove that the points K, L, M, N lie on a circle.



Proof. Let O be the center of the spiral similarity $\mathcal{S}$ that sends BD to CA. Then, by Property 18.5, we get that $\mathcal{S} : N \to M$, i.e. $\mathcal{S} : BN \to CM$. Now, by Property 18.2, since $BN \cap CM = K$, we get that $O = (KBC) \cap (KNM)$, i.e. $ONKM$ is cyclic.

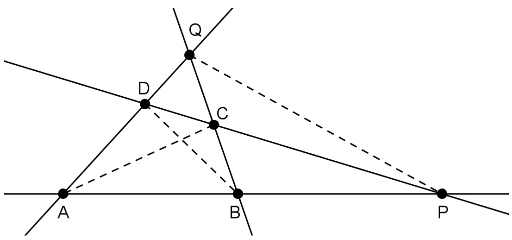
By Property 18.2, since $\mathcal{S} : BD \to CA$ and $BD \cap CA = K$, we get that $O = (KBC) \cap (KDA) \neq K$, i.e. $O \in (KBC)$ and $O \in (KDA)$. Now, the lines AD, BC and KO are concurrent as pairwise radical axes of $(ABCD)$, $(BCKO)$ and $(DAOK)$. But also, the lines AD, BC and ML are concurrent as pairwise radical axes of $(ABCD)$, $(BCML)$ and $(DALM)$. Therefore, $E = AD \cap BC \cap LM \cap OK$. We finish by Intersecting Secants Theorem $\overline{EM} \cdot \overline{EL} = \overline{ED} \cdot \overline{EA} = \overline{EK} \cdot \overline{EO}$ which implies that $OLMK$ is cyclic. ■

Related problems: 84, 119, 124, 153, 158, 163, 186 and 194.

Chapter 19

Complete quadrilateral

A complete quadrilateral is a system of four lines (no three of which pass through the same point) and the six points of intersection of these lines.



Among the six points of a complete quadrilateral there are three pairs of points that are not already connected by lines. The line segments connecting these pairs are called *diagonals* of the complete quadrilateral.

In all of the following properties, let $ABCD$ be a quadrilateral such that the *rays* AB and DC intersect at P and the *rays* BC and AD intersect at Q.

By taking any three of the four lines of a complete quadrilateral, we can get four triangles. For the complete quadrilateral $ABCDPQ$, those triangles are $\triangle ABQ$, $\triangle BCP$, $\triangle CDQ$ and $\triangle DAP$.

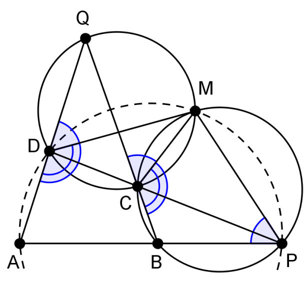
Property 19.1 (Miquel Point)**.** The circumcircles of the four triangles mentioned above pass through a common point, called the Miquel point of the quadrilateral.

Proof 1. Observe that this is a different wording of Property 10.5.2 in the direction when it is given that the points are collinear, which we already proved. ■

Proof 2. Let M be the second intersection of (BCP) and (CDQ). Then,

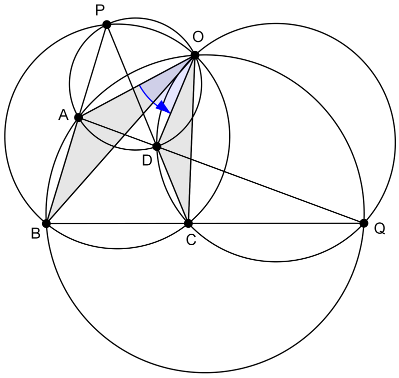
$$\begin{aligned}\angle APM \equiv \angle BPM = 180^\circ - \angle BCM = \\ = \angle QCM = \angle QDM = \\ = 180^\circ - \angle ADM\end{aligned}$$

Therefore, $ADMP$ is cyclic, i.e. $M \in (DAP)$. In exactly the same manner, we can prove that $M \in (ABQ)$. ■

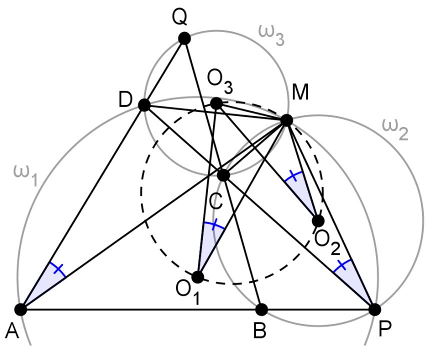


Property 19.2. The Miquel point of a complete quadrilateral $ABCD$ is the center of the spiral similarity $\mathcal{S}_1 : AB \to DC$ and consequently, the center of $\mathcal{S}_2 : AD \to BC$.

Proof. Let O be the center of $\mathcal{S}_1$. Then, by Property 18.2, if $AB \cap DC = P$, then O lies on the circles (PAD) and (PBC). By Property 18.4, if $AD \cap BC = Q$, then O lies on the circles (QAB) and (QDC). From Property 19.1, we know that these four circles concur at the Miquel point of the complete quadrilateral $ABCDPQ$. By Property 18.3, $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same center. ■



Property 19.3. The circumcenters of the four triangles mentioned above, and the Miquel point are concyclic.



Proof. Let M be the Miquel point of $ABCDPQ$. Let the circles $(DAPM)$, $(BCMP)$, $(CDQM)$ and $(ABMQ)$ be ω_1, ω_2, ω_3 and ω_4, respectively and let O_i be the center of ω_i. We will firstly prove that $O_1O_2MO_3$ is cyclic.

MD is the radical axis of ω_1 and ω_3, so O_1O_3 is the bisector of MD and therefore the angle bisector of $\angle MO_1D$. Similarly, O_2O_3 is the bisector of MC and the angle bisector of $\angle MO_2C$.

$$\angle MO_1O_3 = \frac{\angle MO_1D}{2} \overset{\omega_1}{=} \angle MAD \overset{\omega_1}{=} \angle MPD \equiv \angle MPC \overset{\omega_2}{=} \frac{\angle MO_2C}{2} = \angle MO_2O_3$$

Therefore, the quadrilateral $O_1O_2MO_3$ is cyclic. Similarly, $O_2MO_3O_4$ is cyclic. ■

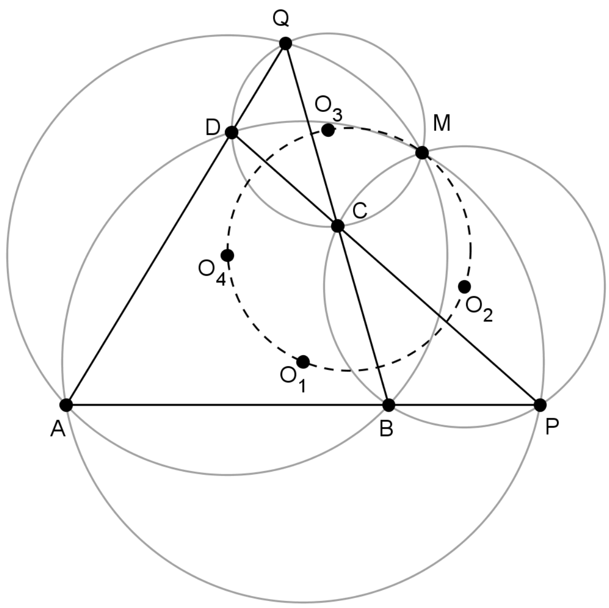
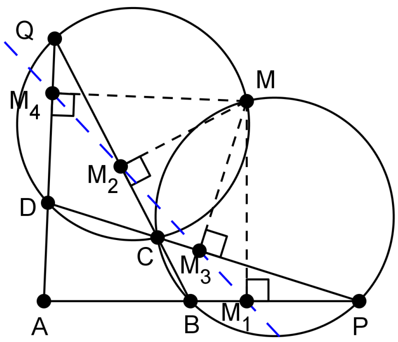


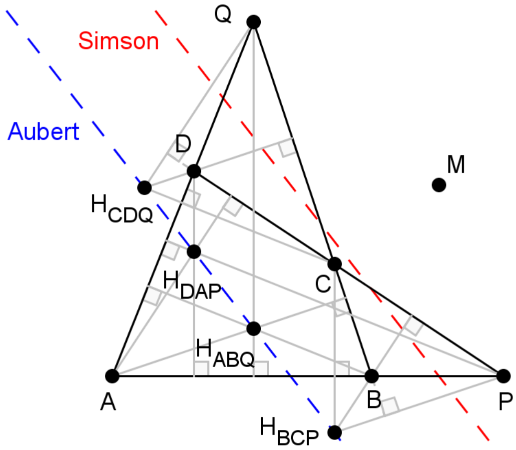
Figure 19.1: The circumcircles of $\triangle ABQ$, $\triangle BCP$, $\triangle CDQ$ and $\triangle DAP$ pass through the Miquel point M. Their circumcenters and M are concyclic.

Property 19.4 (Simson's Line)**.** The feet of the perpendiculars from the Miquel point to the sides of the complete quadrilateral lie on a line, called the *Simson's line* of the complete quadrilateral.



Proof. Let the feet of the perpendiculars from M to AB, BC, CD and DA be M_1, M_2, M_3 and M_4. Using the Simson Line Theorem for $\triangle PBC$ and the point M which lies on its circumcircle, we get that M_1, M_2 and M_3 are collinear. Similarly, by using the Simson Line Theorem for $\triangle CQD$ and the point M, we get that the points M_2, M_3 and M_4 are collinear. ■

Property 19.5 (Aubert's Line)**.** The orthocenters of the four triangles mentioned above lie on a line, called the *Aubert's line*, which is parallel to the Simson's line of the complete quadrilateral.



Proof. By Property 10.6.2, we know that the Simson line from M bisects the line segment MH, where H is the orthocenter of the triangle. Therefore, the homothety $\mathcal{X}_{M,2}$ sends the Simson line of a triangle, to a line through its orthocenter which is parallel to the Simson line. Since in Property 19.4, we proved that the Simson lines of all four triangles coincide, we get that the orthocenters of all four triangles lie on a line parallel to Simson's line. ■

Property 19.6. The three circles with diameters the diagonals of the complete quadrilateral have a common chord.

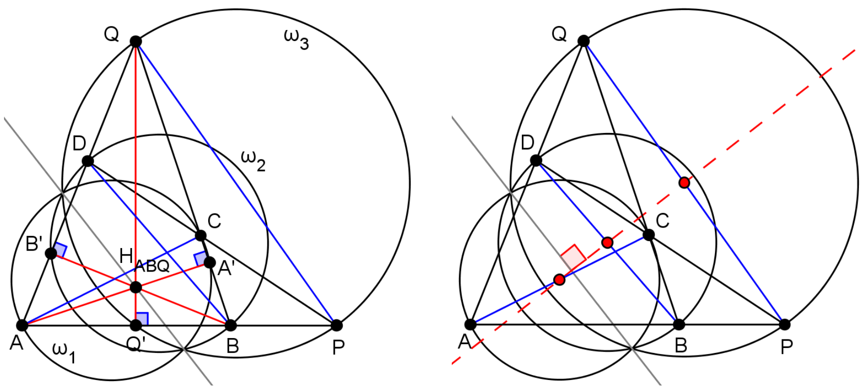
Proof. We will prove that the common chord lies on Aubert's line. Let ω_1, ω_2, and ω_3 be the circles with diameters the diagonals AC, BD and PQ, respectively. Let $\omega_1 \cap BQ = A'$, $\omega_2 \cap AQ = B'$ and $\omega_3 \cap AB = Q'$. Since the inscribed angles over the diameter are right angles, we get that $AA' \perp BQ$, $BB' \perp AQ$ and $QQ' \perp AB$. Therefore, AA', BB' and QQ' pass through the orthocenter of $\triangle ABQ$, H_{ABQ}. From Property 6.7, we know that

$$\overline{AH_{ABQ}} \cdot \overline{H_{ABQ}A'} = \overline{BH_{ABQ}} \cdot \overline{H_{ABQ}B'} = \overline{QH_{ABQ}} \cdot \overline{H_{ABQ}Q'},$$

which is equivalent to

$$pow(H_{ABQ}, \omega_1) = pow(H_{ABQ}, \omega_2) = pow(H_{ABQ}, \omega_3).$$

Therefore, H_{ABQ} has equal powers to all three circles. Similarly, we can get that the other three orthocenters also have equal powers to the three circles. Therefore, there isn't a single radical center of the three circles, but all the points on the line containing the orthocenters have equal powers to all three circles. Therefore, Aubert's line is the common chord of the circles with diameters the diagonals of the complete quadrilateral. ■



Property 19.7 (Gauss' Line)**.** The midpoints of the diagonals of the complete quadrilateral lie on a line, called the *Gauss' line*, which is perpendicular to Simson's and Aubert's line.

Proof. Since the circles ω_1, ω_2, and ω_3 defined in Property 19.6 have a common chord and their centers are the midpoints of the diagonals of the complete quadrilateral, we can conclude that the midpoints of the diagonals are collinear. We also know that the line joining the centers of two circles is perpendicular to their common chord (Property 14.3), thus Gauss's Line is perpendicular to Aubert's Line. ■

19.1 Cyclic Quadrilateral

Property 19.8. The Miquel point of $ABCD$ lies on the line PQ if and only if $ABCD$ is cyclic.

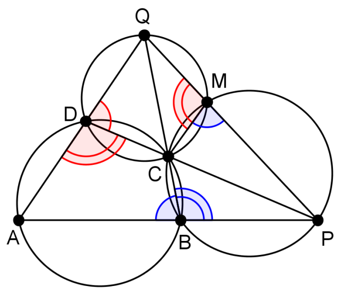
Proof. Let M be the Miquel point of the quadrilateral $ABCD$. Then $MCBP$ and $MQDC$ are cyclic quadrilaterals.

$$\angle PMC = 180^\circ - \angle PBC = \angle ABC$$

$$\angle QMC = 180^\circ - \angle QDC = \angle ADC$$

$$\therefore \angle PMC + \angle QMC = \angle ABC + \angle ADC$$

The Miquel point M lies on PQ iff the left-hand side is 180°. The right-hand side is 180° iff $ABCD$ is a cyclic quadrilateral. ■



Property 19.9. Let $ABCD$ be a cyclic quadrilateral, inscribed in a circle ω centered at O. Let the intersection of the diagonals AC and BD be R. Let M be the Miquel point of $ABCD$. Then,

- The point M lies on the circumcircles of $\triangle AOC$ and $\triangle BOD$;
- The line OM is perpendicular to the line PQ;
- The points O, R, M are collinear.

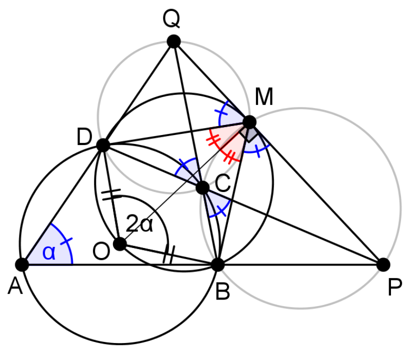
Proof.

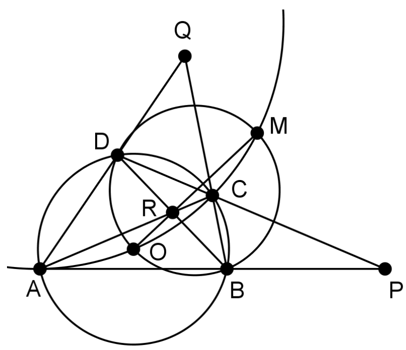
$$\angle BOD = 2\angle BAD = 2\alpha$$

$$\angle DMQ = \angle DCQ = 180^\circ - \angle DCB = \angle BAD = \alpha$$

Similarly, $\angle BMP = \alpha$, so $\angle DMB = 180^\circ - 2\alpha$. Now, $\angle BOD + \angle BMD = 180^\circ$, so $M \in (BOD)$. Similarly, $M \in (AOC)$. □

From the cyclic quadrilateral $OBMD$, since $\overline{OB} = \overline{OD}$, we get that $\angle OMB = \angle OMD$. Also, $\angle BMP = \alpha = \angle DMQ$. Adding these last two equations side by side, we get that $\angle OMP = \angle OMQ$. Since $ABCD$ is cyclic, by Property 19.8, we know that $M \in PQ$. Therefore, $OM \perp PQ$. □





The pairwise radical axes of the circles $(ABCD)$, $(AOCM)$ and $(BODM)$ are AC, BD and OM, so they are concurrent at the radical center of the three circles. Since $AC \cap BD = R$, we get that $R \in OM$. ■

Property 19.10. Let S be the second intersection of (ABR) and (CDR). Let T be the second intersection of (BCR) and (DAR). Then,

- The points $P - R - S$ and $Q - R - T$ are collinear;
- The quadrilaterals $BCOS$, $DAOS$, $ABOT$ and $CDOT$ are cyclic;
- The points $Q - S - O$ and $P - T - O$ are collinear;
- The quadrilaterals $ACPS$, $BDPS$, $ACQT$ and $BDQT$ are cyclic;
- The quadrilaterals $PMOS$, $QMRS$, $QMOT$ and $PMRT$ are cyclic;
- The point O is the orthocenter of $\triangle PQR$.

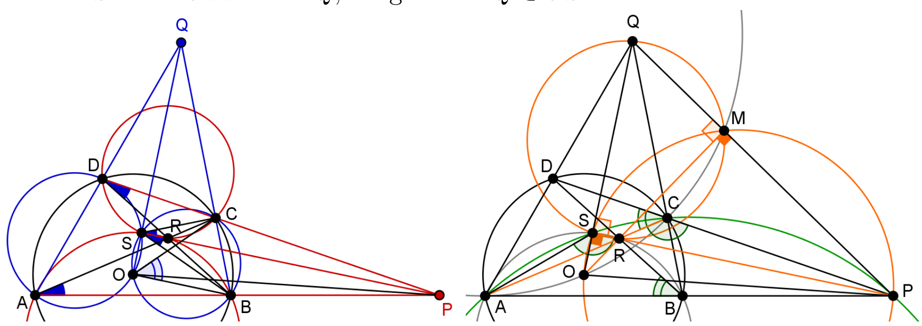
Proof. We will prove the properties only for the point S. The ones for T are completely analogous.

The pairwise radical axes of the circles $(ABCD)$, $(ABRS)$ and $(CDRS)$ are AB, CD and RS, so they are concurrent at the radical center of the three circles. Since $AB \cap CD = P$, we get that $P \in RS$. □

$$\angle BSC = \angle BSR + \angle RSC = \angle BAR + \angle RDC \equiv \angle BAC + \angle BDC = 2\angle BAC$$

But as central angle $\angle BOC = 2\angle BAC$, so $\angle BSC = \angle BOC$, i.e. $S \in (BCO)$. Similarly, $S \in (DAO)$. □

The pairwise radical axes of the circles $(ABCD)$, $(BCOS)$ and $(DAOS)$ are BC, DA and OS, so they are concurrent at the radical center of the three circles. Since $BC \cap DA = Q$, we get that $Q \in OS$. □



$$\angle ASP \equiv \angle ASR = 180^\circ - \angle ABR \equiv 180^\circ - \angle ABD = 180^\circ - \angle ACD = \angle ACP$$

Thus, $S \in (ACP)$. Similarly, $S \in (BDP)$. □

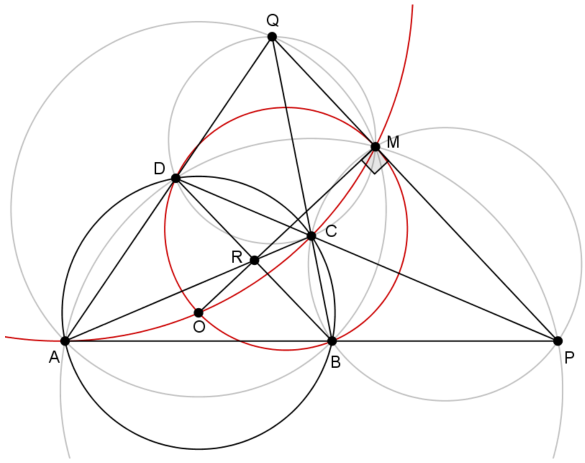
From the Intersecting Chords Theorem in $(AOCM)$ and $(ASCP)$, we get

$$\overline{OR} \cdot \overline{RM} = \overline{AR} \cdot \overline{RC} = \overline{PR} \cdot \overline{RS}$$

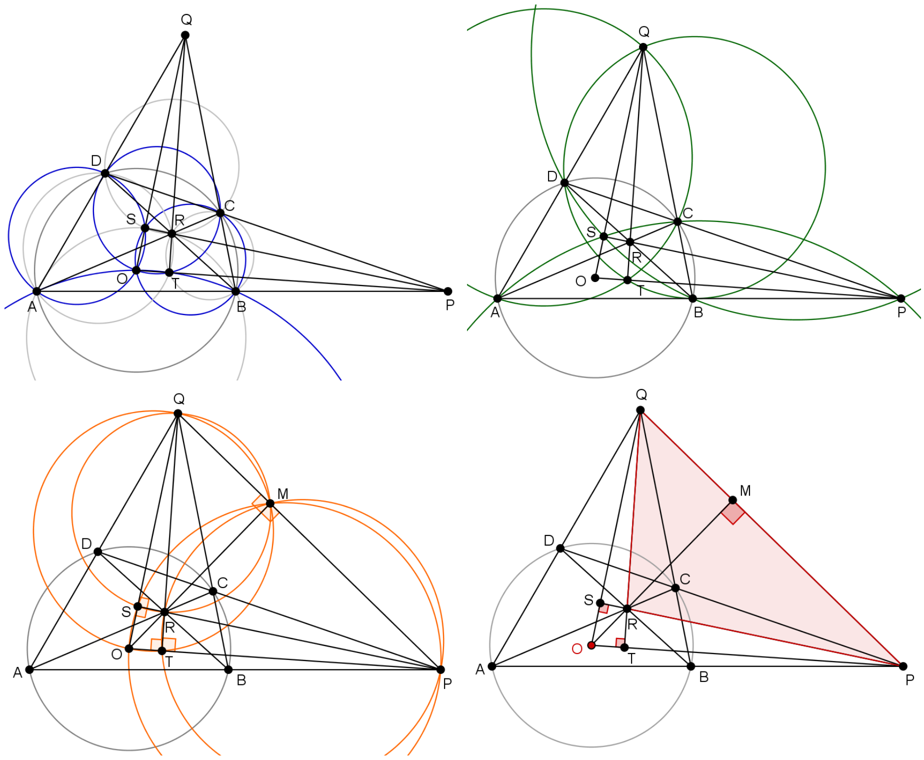
and therefore $PMSO$ is cyclic. Therefore, $\angle OSP = \angle OMP = 90^\circ$. Now, $\angle QSR + \angle QMR = 90^\circ + 90^\circ = 180^\circ$, so $QMRS$ is also cyclic. □

Finally, since $RO \perp PQ$ and $QO \perp PR$, we get that O is the orthocenter of $\triangle PQR$. ■

Here is a complete visualisation of the properties we proved in Property 19.9.

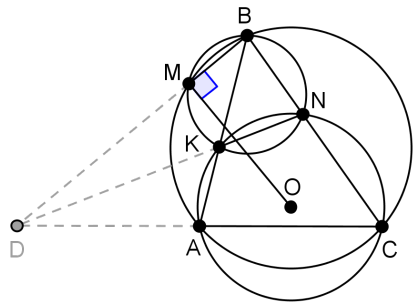


Here is a complete visualisation of the properties we proved in Property 19.10.



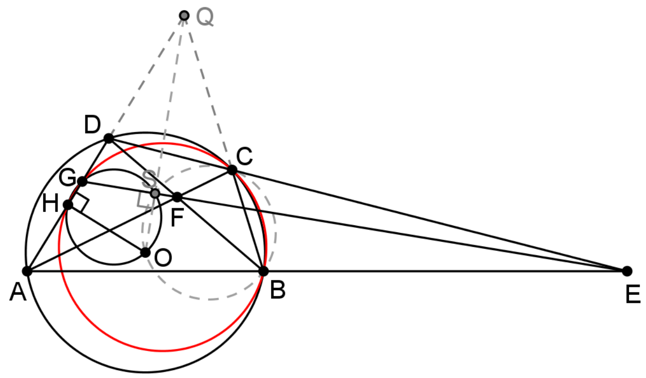
It is very important to learn to recognize these configurations because they show up in many olympiad problems. However, the configuration is not always complete, so sometimes you have to draw additional points, lines or circles in order to come to these "well-known" configurations.

Example 19.1 (IMO 1985/5)**.** A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.



Proof. Let $AC \cap KN = D$. Let's take a look at the complete quadrilateral $ACNKBD$. The triangles $\triangle ACB$ and $\triangle KNB$ are two of the four triangles formed by the lines of the complete quadrilateral, so their circumcircles intersect at the Miquel Point of the complete quadrilateral, i.e. M is the Miquel point of $ACNKBD$. Since $ACNK$ is cyclic, by Property 19.8, $M \in BD$. Finally, by Property 19.9, we get that $OM \perp BD$, i.e. $\angle OMB = 90°$. ■

Example 19.2. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let lines AB and CD meet at E, and let AC and BD meet at F. Furthermore, let EF and AD meet at G, and let H be the projection of O onto AD. Prove that $BCGH$ is cyclic.



Proof. In order to complete the well-know configuration, let $AD \cap BC = Q$ and $OQ \cap EF = S$. From Property 19.10 we know that $S \in (BOC)$ and $OQ \perp EF$. Now, $\angle OHG + \angle OSG = 90° + 90° = 180°$, so $OSGH$ is cyclic. From the Intersecting Secants Theorem for $(OSGH)$ and $(BCSO)$, we get $\overline{QG} \cdot \overline{QH} = \overline{QS} \cdot \overline{QO} = \overline{QC} \cdot \overline{QB}$, which means that $GHBC$ is cyclic. ■

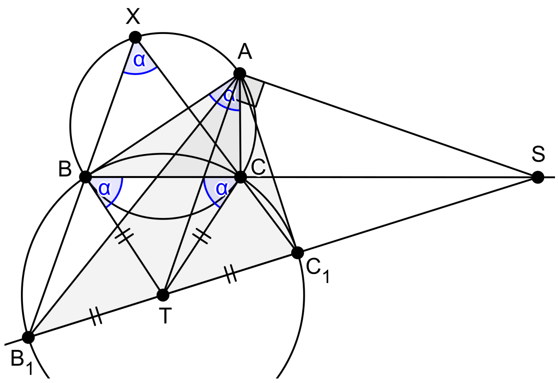
Example 19.3 (USA TST 2007)**.** Acute triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $\overline{B_1T} = \overline{BT} = \overline{C_1T}$. Prove that triangles ABC and AB_1C_1 are similar.

Proof. Let $X = B_1B \cap C_1C$. First of all, we will prove that $X \in \omega$.

Since TB and TC are tangents to (ABC), we get $\angle TBC = \angle BAC = \angle BCT$. Now, $\angle BTC = 180° - \angle TBC - \angle TCB = 180° - 2\alpha$.

$$\begin{aligned}\angle BXC = \angle B_1XC_1 &= 180° - (\angle XB_1C_1 + \angle XC_1B_1) = \\ &= 180° - (\angle BB_1T + \angle CC_1T) = \\ &= 180° - \left(\frac{180° - \angle B_1TB}{2} + \frac{180° - \angle C_1TC}{2}\right) = \\ &= 180° - \left(\frac{360° - (\angle B_1TB + \angle C_1TC)}{2}\right) = \\ &= 180° - (180° - \frac{180° - \angle BTC}{2}) = \\ &= \frac{180° - (180° - 2\alpha)}{2} = \frac{2\alpha}{2} = \alpha = \angle BAC\end{aligned}$$

Therefore, X lies on ω. Now, let's take a look at the complete quadrilateral



B_1C_1CBXS. We want to prove that A is its Miquel point. From Property 19.1, we know that the Miqeul point lies on (XBC) and from Property 19.9 we know that it is the foot of the perpendicular from the center T to the third diagonal XS. We can see that A is the unique point outside (B_1C_1CB) such that $AS \perp AT$ and $A \in (XBC)$. Therefore, A must be the Miquel point of the complete quadrilateral, so, by Property 19.2, it is the center of the spiral similarity that sends BC to B_1C_1 and thus $\triangle ABC \sim \triangle AB_1C_1$. ■

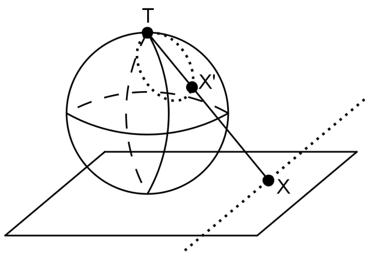
Related problems: 131, 194, 199 and 202.

Chapter 20

Inversion

Inversion, like homothety, is a function that sends a point to another point. However, before continuing, let's firstly introduce the term "extended plane", because inversion is defined there.

The *extended plane* is a set of all the points in a plane together with one special point that we will call *the point at infinity* (P_∞). We also imagine that all the lines pass through this point. To make this a little bit clearer, let's imagine a sphere sitting on a horizontal plane. Let the top-most point of the sphere be T. Then, for every point X on the plane, the line TX will intersect the sphere at a unique point X'. If we move a point X on a line, the image points X' on the sphere will make a circle. However, as we go towards infinity on *any* line on the plane, the image circle will pass through the *same* image, i.e. the point T. Thus, in our scenario, it is OK to imagine that all the lines pass through the same point at infinity.

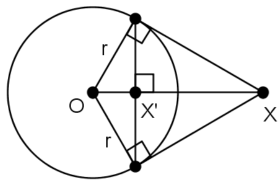


Now, back to the formal definition of inversion. Inversion with center O and radius r is a function on the extended plane that sends a point X to a point X' on the ray OX, such that $\overline{OX} \cdot \overline{OX'} = r^2$. If X is the center of inversion, then it is sent to the point at infinity and vice versa. The circle with center O and radius r is called the *circle of inversion.* From the definition, we can easily check that $(X')' = X$.

$$\mathcal{J}_{O,r} : X \leftrightarrow \begin{cases} X' \in OX, \overline{OX} \cdot \overline{OX'} = r^2 & O \neq X \neq P_\infty \\ P_\infty & X \equiv O \\ O & X \equiv P_\infty \end{cases}$$

A point

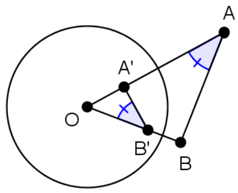
It is easy to see that if a point lies on the circle of inversion, since $\overline{OX} = r$, the image will be the same point, i.e. $X' \equiv X$. If a point is inside the circle of inversion, then its image will be outside and vice versa. How can we construct those images?



Well, if the point X is outside the circle of inversion, we draw the tangents from X to the circle of inversion. The image X' is found as the intersection of the line connecting the tangent points and the line OX. It can be easily checked, by similarity of triangles, that the equation in the definition is satisfied. Since $(X')' \equiv X$, it can be easily figured out what the image is if X is inside the circle of inversion.

Two points

Let A' and B' be the images of the points A and B under inversion with center O and radius r.



Then, $\overline{OA} \cdot \overline{OA'} = r^2 = \overline{OB} \cdot \overline{OB'}$.

$$\therefore \frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OB'}}{\overline{OA'}}$$

Keeping in mind that $O - A - A'$ and $O - B - B'$ are collinear, i.e. $\angle AOB = \angle A'OB'$, we get that $\triangle OAB \sim \triangle OB'A'$. From this, we conclude two important properties that will be further used when solving problems:

Property 20.1. Let A' and B' be the images of the points A and B under inversion with center O and radius r. Then,

$$\angle OB'A' = \angle OAB \tag{20.1}$$

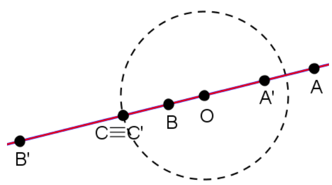
Property 20.2. Let A' and B' be the images of the points A and B under inversion with center O and radius r. Then,

$$\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA} \cdot \overline{OB}} \tag{20.2}$$

Proof. From the similarity $\triangle OAB \sim \triangle OB'A'$, we get

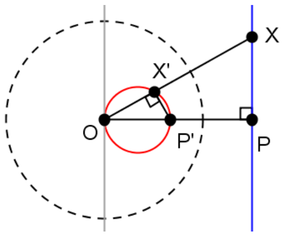
$$\frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{OA'}}{\overline{OB}} = \frac{\overline{OA'} \cdot \overline{OA}}{\overline{OB} \cdot \overline{OA}} = \frac{r^2}{\overline{OA} \cdot \overline{OB}} \qquad \blacksquare$$

A line



A line that passes through the center is sent to itself. (Each point is not sent to itself, obviously, but the line as a figure is sent to itself.)

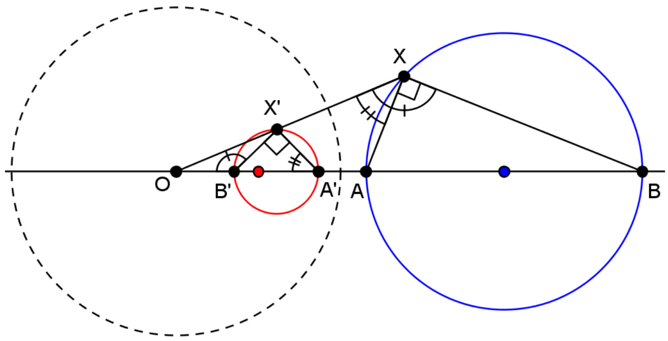
Let's see what happens when a line doesn't pass through the center. Let P be the foot of the perpendicular from the center O to the line, with inverse P', and let X be any point on the line, with inverse X'. Then, by Property 20.1, $\angle OX'P' = \angle OPX = 90°$. Therefore, X' lies on a circle with diameter OP'. In conclusion, a line that doesn't pass through the center is sent to a circle through the center. Moreover, this circle is tangent to the line through O parallel to the original line.



A circle

From the previous case, it's obvious that a circle that passes through the center is sent to a line.

Let's see what happens when a circle doesn't pass through the center. Let A and B be the points on the original circle that are closest and furthest, respectively, to the center of inversion. Then AB is diameter of the original circle. Let A' and B' be the images of A and B, respectively.

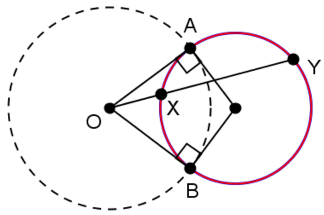


Let X be another point on the original circle. Then $\angle OXB - \angle OXA = \angle AXB$, which is a right angle. By Property 20.1, we get

$$\begin{aligned} \angle OB'X' - \angle OA'X' &= 90° && [\angle OB'X' \text{ is an exterior angle for } \triangle A'B'X'] \\ \angle B'A'X' + \angle A'X'B' - \angle OA'X' &= 90° && [\angle B'A'X' \equiv \angle OA'X'] \\ \angle A'X'B' &= 90° \end{aligned}$$

So, X' lies on a circle with diameter $A'B'$. In conclusion, a circle that doesn't pass through the center of inversion is sent to a circle. Moreover, the center of the original circle and the center of the image circle are collinear with the center of inversion. However, the center of the original cicle *is not sent* to the center of the image circle.

In the case when the original circle is outside the circle of inversion, the image circle is inside and vice versa. When the original circle intersects the circle of inversion, it shares two common points with the image circle (the ones that are on the circle of inversion). So, is it possible, under any conditions, that a circle is sent to itself? Let's assume it is and see if we can understand the conditions when that happens. Let ω be a circle that intersects the circle of inversion at A and B. Let X be any other point on ω and let Y be the second intersection of OX and ω. Since we assumed that ω is sent to itself under the inversion, Y must be the image of X. Therefore, $\overline{OX} \cdot \overline{OY} = r^2$. But A lies on the circle of inversion, so $\overline{OA} = r$. Therefore, $\overline{OX} \cdot \overline{OY} = \overline{OA}^2$, so by the Secant-Tangent Theorem, we get that OA is tangent to ω, i.e. the circle of inversion and ω are orthogonal.



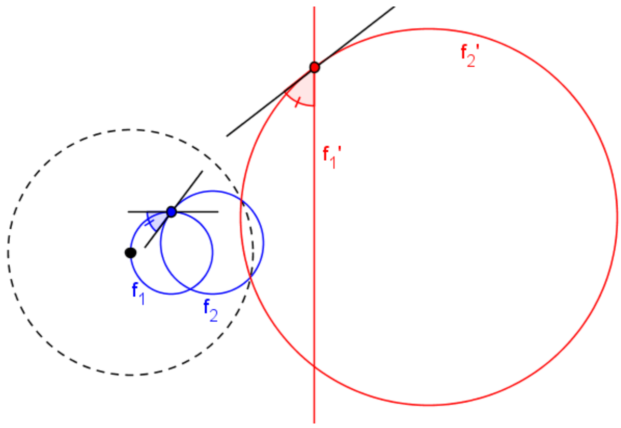
In conclusion, a circle orthogonal to the circle of inversion is sent to itself.

Property 20.3. Let f_1 and f_2 be two figures (line or circle). Let $\mathcal{J} : f_1 \leftrightarrow f_1'$ and $\mathcal{J} : f_2 \leftrightarrow f_2'$. Then, inversion preserves angles between figures[1], i.e.

$$\angle(f_1, f_2) = \angle(f_1', f_2')$$

As a consequence, f_1 is tangent to f_2 if and only if f_1' is tangent to f_2'.

Proof. This can be proven using Property 20.1 in the intersection point. ■

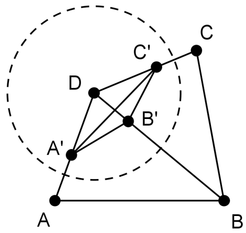


Summary

1. A point on the circle of inversion is sent to itself.
2. A line passing through the center is sent to itself.
3. A line not passing through the center is sent to a circle through the center.
4. A circle passing through the center is sent to a line.
5. A circle not passing through the center is sent to a circle.
    5.1. A circle orthogonal to the circle of inversion is sent to itself.

[1]The angle between two circles is defined as the angle between the tangents to the circles at a point of intersection. Analogously, the angle between a circle and a line is defined as the angle between the line and the tangent to the circle at a point of intersection with the line.

Example 20.1 (Ptolemy's Inequality)**.** Let $ABCD$ be a quadrilateral. Prove that $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}$ and that equality holds iff $ABCD$ is a cyclic quadrilateral.



Proof. Let's invert with center D and any radius r. Let the images of A, B and C be A', B' and C', respectively. By the Triangle Inequality for $\triangle A'B'C'$, we have $\overline{A'B'} + \overline{B'C'} \geq \overline{A'C'}$. By Property 20.2, this is equivalent to

$$\overline{AB} \cdot \frac{r^2}{\overline{DA} \cdot \overline{DB}} + \overline{BC} \cdot \frac{r^2}{\overline{DB} \cdot \overline{DC}} \geq \overline{AC} \cdot \frac{r^2}{\overline{DA} \cdot \overline{DC}}.$$

Multiplying by $\overline{DA} \cdot \overline{DB} \cdot \overline{DC}$ and dividing by r^2 on both sides, we get:

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}.$$

Equality holds iff the points A', B' and C' are collinear, i.e. iff the points A, B and C are concyclic with the center of inversion D. ■

Example 20.2 (IMO 1996/2)**.** Let P be a point inside triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D and E be the incenters of $\triangle APB$ and $\triangle APC$, respectively. Show that AP, BD and CE are concurrent.

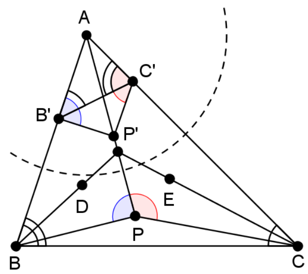
Proof. Here, we see that there are many angles in the condition that are in the form $\angle AXY$ with fixed A. That's why we will try to invert through A.

By Property 20.1, the condition now becomes

$$\angle AB'P' - \angle AB'C' = \angle AC'P' - \angle AC'B'$$

$$\angle P'B'C' = \angle P'C'B'$$

$$\therefore \overline{P'B'} = \overline{P'C'} \qquad (*)$$



If we want to prove that the angle bisectors BD and CE intersect the line segment AP at the same point, then by the Angle Bisector Theorem, we need to prove that

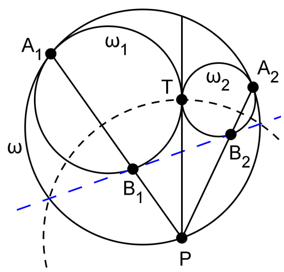
$$\frac{\overline{AB}}{\overline{BP}} = \frac{\overline{AC}}{\overline{CP}}$$

From the similarity of the triangles $\triangle APB \sim \triangle AB'P'$ and $\triangle ACP \sim \triangle AP'C'$ we get

$$\frac{\overline{AB}}{\overline{BP}} = \frac{\overline{AP'}}{\overline{P'B'}} \overset{(*)}{=} \frac{\overline{AP'}}{\overline{P'C'}} = \frac{\overline{AC}}{\overline{CP}}$$ ■

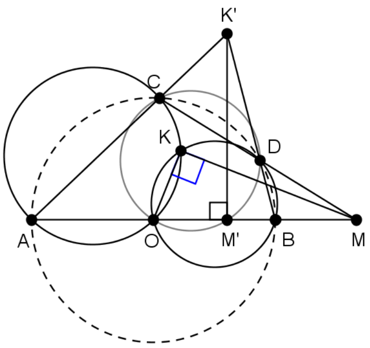
Example 20.3. Two circles ω_1 and ω_2 touch each other externally at T. They also touch a circle ω internally at A_1 and A_2, respectively. Let P be one point of intersection of ω with the common tangent to ω_1 and ω_2 at T. The line PA_1 meets ω_1 again at B_1 and the line PA_2 meets ω_2 again at B_2. Prove that B_1B_2 is a common tangent to ω_1 and ω_2.

Proof. In many problems, we tend to invert through a "busy" point, i.e. a point through which many lines or circles pass. Consider the inversion $\mathcal{J}(P, \overline{PT})$. In this way, since PT is tangent to ω_1, i.e. ω_1 is orthogonal to the circle of inversion, $\mathcal{J} : \omega_1 \leftrightarrow \omega_1$. Let's see what is A_1 sent to. The image of A_1 must be on PA_1. Also, since A_1 lies on ω_1, then the image of A_1 must lie on the image of ω_1. Therefore, $\mathcal{J} : A_1 \leftrightarrow B_1$. Similarly, $\mathcal{J} : \omega_2 \leftrightarrow \omega_2$ and $\mathcal{J} : A_2 \leftrightarrow B_2$. Now, the circle ω passes through the center of inversion P, so it will be sent to a line. The points A_1 and A_2 lie on this circle, so their images will lie on the image line. Therefore, $\mathcal{J} : \omega \leftrightarrow B_1B_2$. Finally, since ω is tangent to ω_1 and ω_2, then by Property 20.3, its image will be tangent to their images, i.e. B_1B_2 will be tangent to ω_1 and ω_2. ■



Example 20.4. A semicircle with diameter AB is centered at O. A line intersects the semicircle at C and D and the line AB at M, such that $\overline{MB} < \overline{MA}$ and $\overline{MD} < \overline{MC}$. Let K be the second point of intersection of the circumcircles of $\triangle AOC$ and $\triangle BOD$. Prove that $\angle MKO = 90°$.

Proof. The point O is one of the busy points in this diagram and also the angle $\angle MKO$ which is of interest for us is in the form $\angle OXY$, so it is wise to try to invert through O. Consider the inversion $\mathcal{J}(O, \overline{OA})$. By Property 20.1, we need to prove that $\angle OM'K' = 90°$. The points A, B, C and D are sent to themselves since they lie on the circle of inversion. The circles (OAC) and (OBD) pass through the center of inversion, so they are sent to the lines AC and BD, respectively. Since K lies on these circles, then K' must lie on their images, i.e. $AC \cap BD = K'$. The line AB passes through the center of inversion, so it is sent to itself. The line CD doesn't pass through the center, so it is sent to the circle (OCD). Since M lies on the lines AB and CD, its image M' will lie on their images, i.e. $M' = AB \cap (OCD)$. Now, since AB is the diameter of $(ABCD)$, C and D are feet of the altitudes in $\triangle ABK'$ and O is a midpoint in the same triangle. So, (OCD) is the nine point circle of $\triangle ABK'$. Since M' lies on AB and the nine point circle, then it must be the feet of the altitude from K' to AB and therefore $\angle OM'K' = 90°$. ■



Related problems: (Inversion) 61, 88, 180, 201 and 203.

In the following sections, we will present two special types of inversion, commonly used in some Olympiad problems. These sections are written by our guest writer, Nikola Danevski.

20.1 $\sqrt{bc}$ Inversion

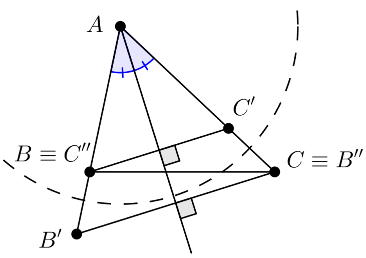
The first one, which we will call *"$\sqrt{bc}$ inversion"* and denote by Ψ, is an inversion centered at A with radius of exactly $\sqrt{bc}$ followed by a reflection with respect to the A-angle bisector. More formally, let $\mathcal{J}_{A,\sqrt{\overline{AB}\cdot\overline{AC}}}$ be the aforementioned inversion and let Φ_{ℓ_α} be the reflection with respect to the angle bisector of α. Then,

$$\Psi = \Phi_{\ell_\alpha} \circ \mathcal{J}_{A,\sqrt{\overline{AB}\cdot\overline{AC}}}.$$

It can be easily checked that if Ψ sends X to X', then it sends X' to X, so we will use the following notation $\Psi : X \leftrightarrow X'$.

Now, let's see why this inversion is useful by seeing the images of some well-known points, lines and circles.

Property 20.1.1. In a $\triangle ABC$, $\Psi : B \leftrightarrow C$.

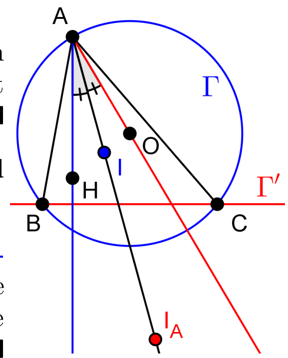


Proof. Let $\mathcal{J} : B \leftrightarrow B'$ and $\mathcal{J} : C \leftrightarrow C'$. Then, by the definition of inversion, we have

$$\overline{AB}\cdot\overline{AB'} = \overline{AC}\cdot\overline{AC'} = r^2 = \overline{AB}\cdot\overline{AC}$$

So, $\overline{AB'} = \overline{AC}$ and $\overline{AC'} = \overline{AB}$ meaning that when B' and C' are reflected about the A-angle bisector, they map into C and B, respectively. ∎

Property 20.1.2. In a $\triangle ABC$, $\Psi : (ABC) \leftrightarrow BC$.



Proof. Since $\Psi : B \leftrightarrow C$ and a circle passing through the center of inversion is sent to a line, we have that $\Psi : (ABC) \leftrightarrow BC$. ∎

Property 20.1.3. If I and I_A are the incenter and $A-$excenter of $\triangle ABC$, respectively, then $\Psi : I \leftrightarrow I_A$.

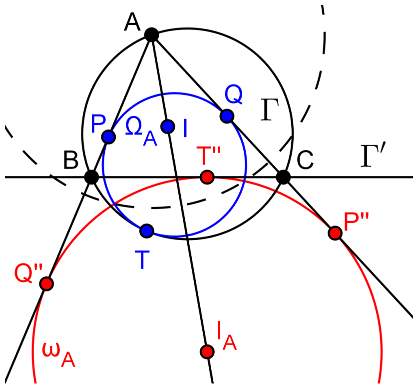
Proof. We know that $\overline{AI}\cdot\overline{AI_A} = \overline{AB}\cdot\overline{AC} = r^2$ (Property 7.5). Since both of these points lie on the A-angle bisector, the reflection with respect to it does not change anything, i.e. $\Psi : I \leftrightarrow I_A$. ∎

Property 20.1.4. If O and H are the circumcenter and orthocenter of $\triangle ABC$, respectively, then $\Psi : AO \leftrightarrow AH$.

Proof. AO is a line passing through the center so it is sent to itself. However, after the reflection through the A-angle bisector it is sent to its isogonal line, which from Property 6.9 we know is AH. ∎

Remark. In general, any line passing through A is sent to its isogonal line.

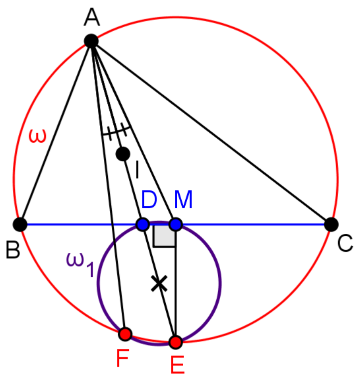
Property 20.1.5. If Ω_A is the A-mixtilinear incircle and ω_A is the A-excircle of $\triangle ABC$, then $\Psi : \Omega_A \leftrightarrow \omega_A$.



Proof. The mixtilinear incircle is tangent to AB, AC and (ABC). Since inversion preserves tangency, the image of the mixtilinear circle will be tangent to their images, which are AC, AB and BC, respectively. Since Ω_A does not pass through the center of inversion A, it will be sent to a circle. Now, WLOG $\overline{AB} \leq \overline{AC}$, let the tangent point of Ω_A with AB be P. Then since Ω_A is "internally" tangent to AB, we get $\overline{AP} < \overline{AB}$. Let $\mathcal{J} : P \leftrightarrow P'$ and $\Psi : P \leftrightarrow P''$. Since $\overline{AX} \cdot \overline{AX'} = r^2 = const.$ for any point X, then $\overline{AP'} > \overline{AB'}$ and thus $\overline{AP''} > \overline{AB''} = \overline{AC}$, i.e. the image of Ω_A is "externally" tangent to AC, so it cannot be the incircle, and must be the $A-$excircle. ■

Now, let's solve a few problems using these properties.

Example 20.1.1 (Russia 2009)**.** Let ABC be a triangle with circumcircle ω. Let the A-angle bisector intersect BC at D and ω again at E. Circle ω_1 with diameter DE intersects ω again at F. Prove that AF is the A-symmedian in $\triangle ABC$.



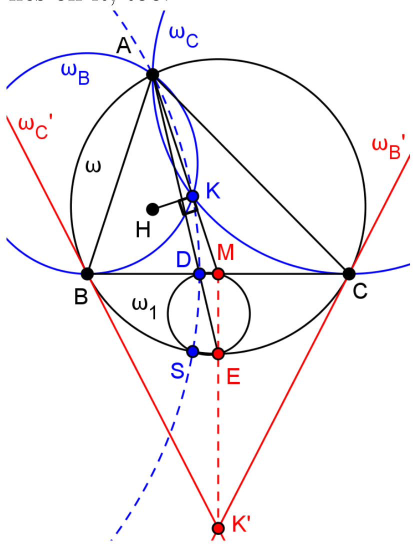
Proof. Let M denote the midpoint of BC. We know that E is the midpoint of the arc $\widehat{BC}$ not containing A, so $\angle EMD = 90°$ and so $M \in \omega_1$. Now let Ψ be the "$\sqrt{bc}$ inversion" and let $\Psi : X \leftrightarrow X'$ for any object X. Since both D and E lie on the A-angle bisector and $D \in BC$ while $E \in (ABC)$, because of Property 20.1.2, we deduce that $\Psi : D \leftrightarrow E$. Now, even though the center of

ω_1 is mapped into a different point under this inversion, it is still on the angle bisector (a property of inversion). Therefore, ω_1' is a circle centered at AI that passes through $D' \equiv E$ and $E' \equiv D$. So $\Psi : \omega_1 \leftrightarrow \omega_1$, i.e. it is fixed under this inversion.

Now let's find the image of the point M. We have $M = \omega_1 \cap BC \neq D$, so $M' = \omega_1' \cap \{BC\}' \neq D'$, i.e. $M' = \omega_1 \cap (ABC) \neq E$. This means that $M' = F$. Since AM is the median of ABC, after the inversion followed by a reflection through the $A-$ angle bisector, the $A-$median maps into its isogonal line, which is the $A-$symmedian. So, $AM' \equiv AF$ is the $A-$symmedian. ■

Example 20.1.2 (Crux Mathematicorum, 4037)**.** In a non-isosceles triangle ABC, let H and M denote the orthocenter and the midpoint of side BC, respectively. The internal angle bisector of $\angle BAC$ intersects BC and the circumcircle of triangle ABC at points D and $E \neq A$, respectively. If K is the foot of the perpendicular from H to AM and S is the intersection (other than E) of the circumcircles of $\triangle ABC$ and $\triangle DEM$, prove that quadrilateral $ASDK$ is cyclic.

Proof. This problem indeed resembles Example 20.1.1. From there, after applying $\sqrt{bc}$ inversion, we get $\Psi : B \leftrightarrow C$, $\Psi : D \leftrightarrow E$ and $\Psi : S \leftrightarrow M$. Again, let $\Psi : X \leftrightarrow X'$ for any object X. From the properties of inversion, in order to prove that $ASDK$ (which passes through the center of inversion A) is cyclic, we need to prove that $S' - D' - K'$ lie on a line, i.e. we need to prove that $M - E - K'$ are collinear. Since E and M lie on the side bisector of BC, we need to show that K' lies on it, too.

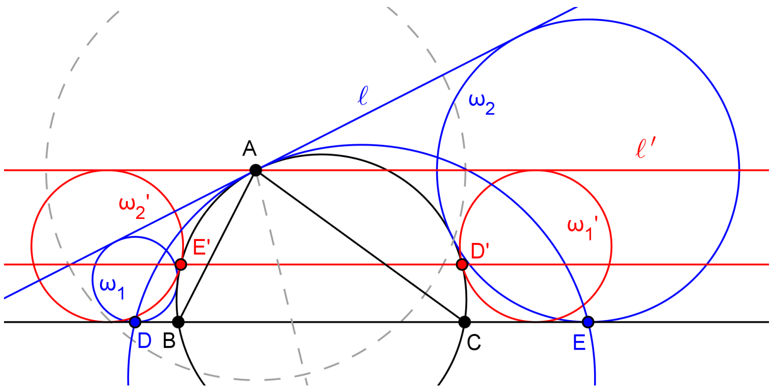


Let us notice that K is in fact the $A-HM$ point of triangle ABC. As such, from Property 10.11.2 we know that it is the intersection of the two circles tangent to BC passing through A and B, and A and C. Let's denote these circles with ω_B and ω_C, respectively. The circle $\omega_B \equiv (ABK)$ which is tangent to BC is sent to a line (since it passes through A) tangent to $\{BC\}'$ and passes through B'. So it is the tangent line to (ABC) through C. Similarly, ω_C is sent to the tangent line to (ABC) through B. So the $A - HM$ point K maps into the intersection of the tangents through B and C of (ABC). Now, since $\overline{K'B} = \overline{K'C}$ as tangent segments, we get that K' lies on the perpendicular bisector of BC. ■

Example 20.1.3. Let ω be the circumcircle of $\triangle ABC$, ℓ be the tangent line to ω at point A. The circles ω_1 and ω_2 are tangent to the lines ℓ, BC and to the circle ω externally. Denote by D and E the points where ω_1 and ω_2 touch BC, respectively. Prove that the circumcircles of $\triangle ABC$ and $\triangle ADE$ are tangent.

Proof. Again, let Ψ be the "$\sqrt{bc}$ inversion" and let $\Psi : X \leftrightarrow X'$ for any object X. From Property 20.1.2, we know that $\Psi : (ABC) \leftrightarrow BC$.

The line ℓ passes through the center of inversion A, so it is sent to a line through A. Also, inversion preserves tangency and since ℓ is tangent to (ABC) at A it follows that ℓ' is "tangent" to BC, meaning they have only one common point, which is the image of A, i.e P_∞. So ℓ' is the line parallel to BC through A.



The circles ω_1 and ω_2 are tangent to ℓ, BC and (ABC), so their images are tangent to ℓ', (ABC) and BC. Since D is the tangency point of ω_1 and BC, it follows that D' is the point of tangency between ω_1' and (ABC) and so D' lies on (ABC). Similarly, E' lies on (ABC). Furthermore, the circles ω_1' and ω_2' are tangent to (ABC) and to the parallel lines ℓ' and BC, so they are symmetric with respect to the perpendicular bisector of BC. Thus, D' and E' are symmetric with respect to the perpendicular bisector of BC as well. So, $E'D'$ is parallel to BC. Therefore, since $\Psi : E'D' \leftrightarrow (AED)$ and $\Psi : BC \leftrightarrow (ABC)$, we get that (AED) and (ABC) are tangent to each other. ■

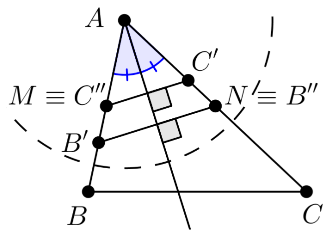
20.2 $\sqrt{\frac{bc}{2}}$ Inversion

This type of inversion is similar to the previous one as it is also followed by a reflection through the A-angle bisector, with the exception that the inversion centered at A is now with radius of $\sqrt{\frac{bc}{2}}$. Again, let $\mathcal{J}_{A,\sqrt{\frac{\overline{AB}\cdot\overline{AC}}{2}}}$ be the aforementioned inversion and let Φ_{ℓ_α} be the reflection with respect to the angle bisector of α. Then,

$$\Psi' = \Phi_{\ell_\alpha} \circ \mathcal{J}_{A,\sqrt{\frac{\overline{AB}\cdot\overline{AC}}{2}}}.$$

It can be easily checked that if Ψ' sends X to X', then it sends X' to X, so we will use the following notation $\Psi' : X \leftrightarrow X'$.

Property 20.2.1. Let M and N be the midpoints of sides AB and AC in a $\triangle ABC$, respectively. Then $\Psi' : B \leftrightarrow N$ and $\Psi' : C \leftrightarrow M$.

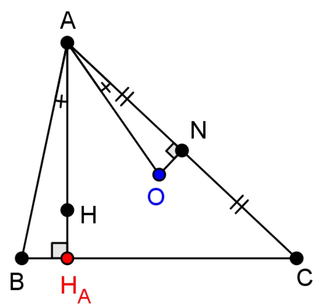


Proof. Let $\mathcal{J} : B \leftrightarrow B'$ and $\Psi' : B \leftrightarrow B''$. Then,

$$\overline{AB} \cdot \overline{AB'} = r^2 = \frac{\overline{AB} \cdot \overline{AC}}{2}, \quad \therefore \overline{AB''} = \overline{AB'} = \frac{\overline{AC}}{2} = \overline{AN}$$

So, when B' is reflected about the $A-$angle bisector, it will coincide with N, i.e. $B'' \equiv N$. Similarly, $\Psi' : C \leftrightarrow M$. ■

Property 20.2.2. Let O be the circumcenter of $\triangle ABC$ and let H_A be the foot of the A-altitude. Then, $\Psi' : O \leftrightarrow H_A$.

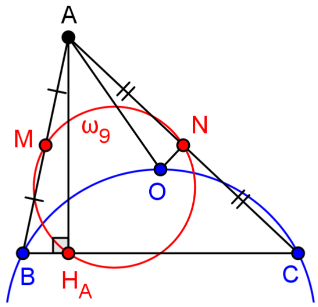


Proof. Let N be the midpoint of side AC. Since O and H are isogonal conjugates (Property 6.9) and since $ON \perp AC$, by AA we get $\triangle AH_AB \sim \triangle ANO$.

$$\therefore \frac{\overline{AH_A}}{\overline{AN}} = \frac{\overline{AB}}{\overline{AO}}, \quad \therefore \overline{AH_A} \cdot \overline{AO} = \overline{AB} \cdot \overline{AN} = \frac{\overline{AB} \cdot \overline{AC}}{2} = r^2$$

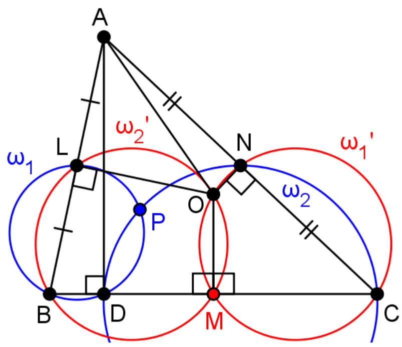
Thus, since the lines AH_A and AO are symmetric with respect to the $A-$ angle bisector, we get $\Psi' : H_A \leftrightarrow O$. ■

Property 20.2.3. Let O be the circumcircle of $\triangle ABC$ and let ω_9 be its nine-point circle. Then, $\Psi' : \omega_9 \leftrightarrow (BCO)$.



Proof. Since (BCO) is a circle not passing through the center of inversion A, it will be sent to a circle. By Property 20.2.1 and Property 20.2.2, the points B and C are sent to the midpoints of AC and AB, respectively, while O is sent to the foot of the A-altitude. Since the midpoints and the feet of the altitudes in any triangle lie on its nine-point circle, we get $\Psi' : (BCO) \leftrightarrow \omega_9$. ■

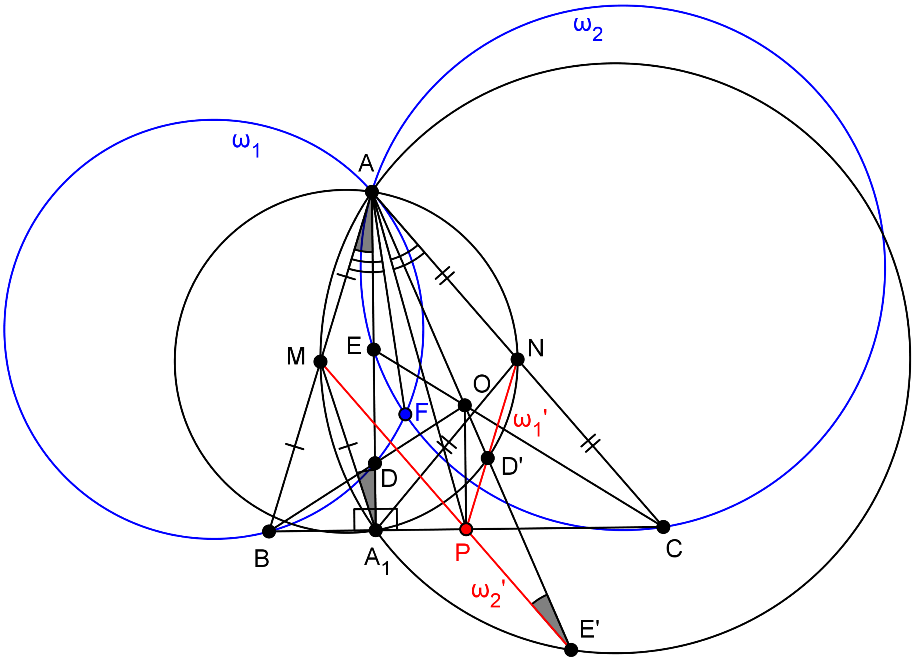
Example 20.2.1. Let L and N be the midpoints of AB and AC, respectively and let D be the projection of A on BC. Circles (BDL) and (CDN) meet again at P. Prove that AP is the A-symmedian.



Proof. In this example we should consider the $\sqrt{\frac{bc}{2}}$ inversion because we have both the midpoints of AB and AC and a claim that a line is a symmedian. Let Ψ' be the "$\sqrt{\frac{bc}{2}}$ inversion". From Property 20.2.1 and Property 20.2.2, we know that $\Psi' : B \leftrightarrow N$, $\Psi' : C \leftrightarrow L$ and $\Psi' : D \leftrightarrow O$, where O is the circumcenter of $\triangle ABC$. Now, (BDL) and (CDN) map into circles since they don't contain the center of inversion A. The circle (BDL) maps into (NOC) while (CDN) maps into (LOB). Since the median and the symmedian are isogonal, we just need to show that the second intersection of (NOC) and (LOB) lies on the A-median. Let M be the midpoint of BC. Then, $\angle ONC + \angle OMC = 90^\circ + 90^\circ = 180^\circ$ so M lies on (NOC). Similarly, M lies on (LOB) and therefore we deduce that M is the second intersection of the two circles. Obviously, M lies on the $A-$median and so we are done. ■

We will now present yet another proof of Example 13.1.

Example 20.2.2 (Macedonia MO 2017, Stefan Lozanovski)**.** Let O be the circumcenter of the acute triangle ABC ($\overline{AB} < \overline{AC}$). Let A_1 and P be the feet of the perpendiculars from A and O to BC, respectively. The lines BO and CO intersect AA_1 in D and E, respectively. Let F be the second intersection point of (ABD) and (ACE). Prove that the angle bisector od $\angle FAP$ passes through the incenter of $\triangle ABC$.



Proof. Let Ψ' be the "$\sqrt{\frac{bc}{2}}$ inversion" and let $\Psi' : X \leftrightarrow X'$ for any object X. We need to prove that $\angle BAF = \angle PAC$, i.e. AF and AP are isogonal, so it is enough to prove that $F' \in AP$. We will prove that $F' \equiv P$.

From Property 20.2.1 and Property 20.2.2, we know that $\Psi' : B \leftrightarrow N$, $\Psi' : C \leftrightarrow M$ and $\Psi' : A_1 \leftrightarrow O$. Therefore, $\Psi' : BO \leftrightarrow (NA_1A)$, $\Psi' : CO \leftrightarrow (MA_1A)$ and $\Psi' : AA_1 \leftrightarrow AO$. Now, since $D = AA_1 \cap BO$, $D' = AO \cap (NA_1A)$. Similarly, since $E = AA_1 \cap CO$, $E' = AO \cap (MA_1A)$. Therefore, since $F = (ABD) \cap (ACE)$, we get that $F' = ND' \cap ME'$. We will prove that $P \in ND'$ and $P \in ME'$.

Since MP is midsegment in $\triangle ABC$, we have $MP \parallel AC$ and therefore $\angle BMP = \angle BAC = \alpha$. On the other hand,

$$\angle BME' = \angle MAE' + \angle ME'A = \angle BAO + \angle MA_1A = \angle BAO + \angle MAA_1 = \\ = 90^\circ - \gamma + \angle BAA_1 = 90^\circ - \gamma + 90^\circ - \beta = \alpha.$$

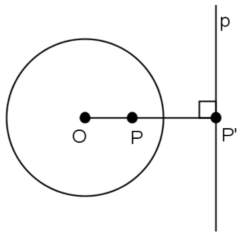
Therefore, $\angle BMP = \angle BME'$, so $P \in ME'$. Similarly, $P \in ND'$. ■

Related problems: ($\sqrt{bc}$ and $\sqrt{\frac{bc}{2}}$ Inversion) 162, 174, 184 and 209.

Chapter 21

Pole & Polar

Let the image of the point P under inversion with respect to the circle with center O and radius r be P'. The *polar* of P is the line p perpendicular to the line OP at P'. In this case, the point P is called the *pole* of p.



We will now present some properties that will be useful when solving problems.

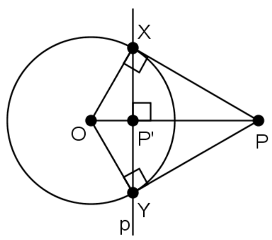
Property 21.1. If P is outside the circle ω, and X and Y are points on ω, such that PX and PY are tangents, then the polar p of P is the line XY.

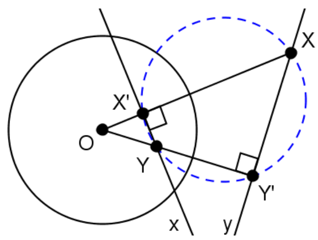
Proof. Recall that the image point P' can be found as the intersection of XY and OP, i.e. $P' \in XY$. By symmetry, $XY \perp OP$. Therefore, by the definition of polar, $p \equiv XY$. ∎

Property 21.2 (La Hire's Theorem)**.** Let x and y be the polars of X and Y, respectively. Then, $X \in y \iff Y \in x$.

Proof. Let X' and Y' be the images of X and Y under the inversion. Then $\overline{OX} \cdot \overline{OX'} = r^2 = \overline{OY} \cdot \overline{OY'}$, which means that the points X, Y, X' and Y' are concyclic. Therefore,

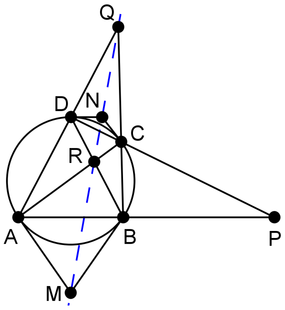
$$X \in y \iff \angle XY'Y = 90^\circ \iff \angle XX'Y = 90^\circ \iff Y \in x \qquad \blacksquare$$

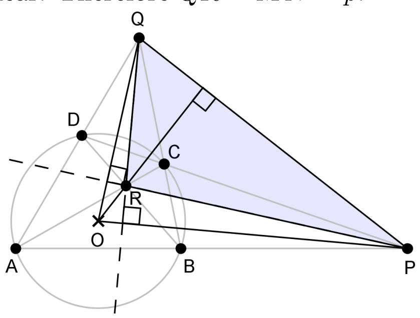




Property 21.3 (Brocard's Theorem)**.** Let $ABCD$ be a cyclic quadrilateral centered at O. Let $AB \cap CD = P$, $BC \cap AD = Q$ and $AC \cap BD = R$. Then, the polar of P is QR. Moreover, the triangle $\triangle PQR$ is autopolar and O is the orthocenter of $\triangle PQR$.

Proof. Let the intersection of the tangents at A and B be M. Then, $AB \equiv m$. Let the intersection of the tangent at C and D be N. Then, $CD \equiv n$. By La Hire's Theorem, since $P \in m$ and $P \in n$, then $M \in p$ and $N \in p$, i.e. $MN \equiv p$. By applying Pascal's Theorem on the points $AACBBD$, we get that $M - R - Q$ are collinear. By applying Pascal's Theorem on the points $CCADDB$, we get that the points $N - R - Q$ are collinear. Therefore $QR \equiv MN \equiv p$.

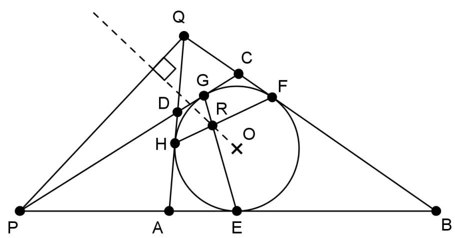




Similarly, we can get that $PR \equiv q$. Also, since R lies on the polars of P and Q, then the polar of R, $r \equiv PQ$. Therefore, the triangle $\triangle PQR$ is autopolar, i.e. the polar of each of the vertices is the opposite side. So, by the definition of polar, it also follows that O is the orthocenter of $\triangle PQR$. ■

We will now solve a few examples to see how these properties of polars can be used in problems. In these examples, we will use lowercase letters to denote the polars of the points in uppercase (e.g. p is the polar of P).

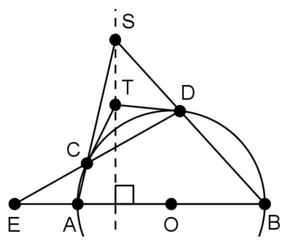
Example 21.1. The quadrilateral $ABCD$ has an inscribed circle ω which is tangent to the sides AB, BC, CD and DA at E, F, G and H, respectively. Let $AB \cap CD = P$, $AD \cap BC = Q$ and $EG \cap FH = R$. If O is the center of ω, then prove that $OR \perp PQ$.



Proof. By Property 21.1, we get $p \equiv EG$ and $q \equiv FH$. Since $R \in p$ and $R \in q$, by La Hire's Theorem, we get $P \in r$ and $Q \in r$, i.e. $r \equiv PQ$. By the definition of polar, $OR \perp r$, i.e. $OR \perp PQ$. ■

Example 21.2. Let AB be a diameter of a semicircle. C and D are two points on the semicircle such that $\overset{\frown}{AC} < \overset{\frown}{AD}$. The tangents to the semicircle at C and D meet at T. If $S = AC \cap BD$, prove that $ST \perp AB$.

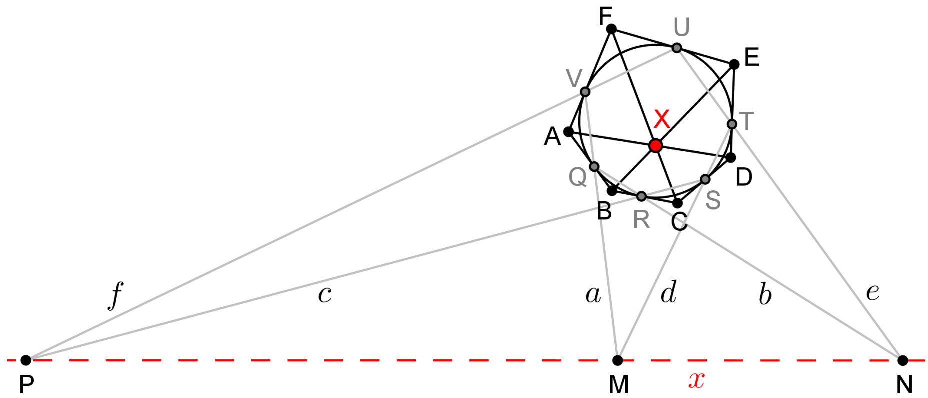
Proof. Let $E = CD \cap AB$. By Property 21.1, $t \equiv CD$. Since $E \in t$, by La Hire's Theorem, $T \in e$. On the other hand, by Property 21.3, $S \in e$. Therefore, $ST \equiv e$, so by the definition of polar $OE \perp ST$, i.e. $AB \perp ST$. ■



Related problems: 131 and 155.

21.1 Pole-Polar Duality

Property 21.4 (Brianchon's Theorem)**.** Let $ABCDEF$ be a hexagon that is circumscribed about a circle ω. Then, its main diagonals, AD, BE and CF, are concurrent.



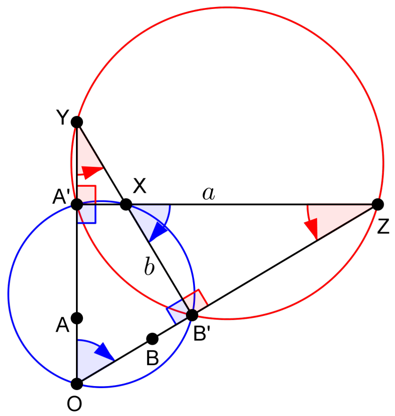
Proof. Let the tangent points of ω with AB, BC, CD, DE, EF, FA be Q, R, S, T, U, V, respectively. Since AQ, AV are tangents to ω, by Property 21.1, we get that $a \equiv QV$. Similarly, $d \equiv ST$. We need to prove that AD, BE, CF are concurrent at some point X, i.e. we need to prove that $X \in AD, BE, CF$. Notice that by La Hire's Theorem, if $M \in a$ and $M \in d$, then $A \in m$ and $D \in m$, i.e. $m \equiv AD$, and again by La Hire's Theorem, that $X \in AD \equiv m$ is equivalent to $M \in x$. Similarly, if $N = b \cap e$ and $P = c \cap f$, we need to prove that $M, N, P \in x$, i.e. we need to prove that M, N, P all lie on some line x (and then the point of concurrency X will be the pole of x). But M, N, P are collinear by Pascal's Theorem for the cyclic hexagon $VQRSTU$. ■

Notice that we can derive Brianchon's Theorem directly from Pascal's Theorem, only by interchanging the words "point" and "line", and making whatever grammatical adjustments that are necessary. Using the notations in the previous diagram, we interchange the *point* A with the *line* a, and similarly for the other points. But also, Brianchon's Theorem states that the *lines* AD, BE, CF pass through a common *point*, while Pascal's Theorem states that the *points* $a \cap d, b \cap e, c \cap f$ lie on a common *line*. This is what is known as **pole-polar duality** (or *polarity*, or *polar reciprocation*), a special version of the general *point-line duality*. That's why Pascal's and Brianchon's Theorem are called *dual* theorems. Also, Desargues' Theorem is *self-dual*, i.e. the implication and its converse are duals of each other. You can read more about this methods in [2], an article that I highly recommend.

We will finish this chapter by solving one example using this method, but firstly we need to prove one property that will help us with angle chasing when using duality.

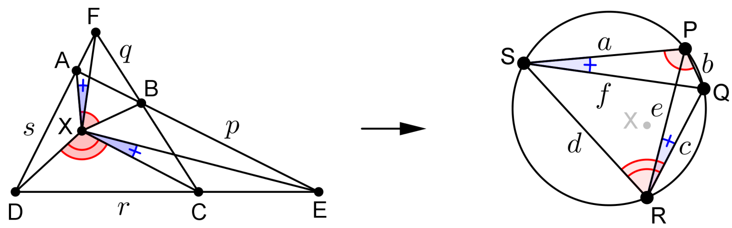
Property 21.5. Let a, b be the polars of A, B, respectively, with respect to a circle with center O and arbitrary radius. Then, using directed angles,

$$\measuredangle(OA, OB) = \measuredangle(a, b) \quad \text{and} \quad \measuredangle(OA, b) = \measuredangle(a, OB)$$



Proof. Let A' and B' be the feet of the perpendiculars from A and B to a and b, respectively. Let $a \cap b = X$, $OA \cap b = Y$ and $OB \cap a = Z$. By the definition of polars, we know that $OA \perp a$ and $OB \perp b$. Therefore, $\angle OA'X = 90° = \angle OB'X$ and $\angle YA'Z = 90° = \angle YB'Z$, so $OA'XB'$ and $A'B'YZ$ are cyclic, from where we get the desired angle equalities. ■

Example 21.3. Let $ABCD$ be a convex quadrilateral. Let $AB \cap CD = E$ and $AD \cap BC = F$. Suppose X is a point inside the quadrilateral such that $\angle AXF = \angle EXC$. Prove that $\angle AXB + \angle CXD = 180°$.



Proof. Let $p \equiv AB$, $q \equiv BC$, $r \equiv CD$ and $s \equiv DA$. We apply duality with center X and arbitrary radius. The lines p, q, r, s are "transformed" to points P, Q, R, S. Since $A = p \cap s$, we get $a \equiv PS$ and similarly $b \equiv PQ$, $c \equiv QR$ and $d \equiv RS$. Also, since $E = p \cap r$, we get $e \equiv PR$ and similarly, $f \equiv QS$. Now, the angle condition $\angle AXF = \angle EXC$, i.e. $\measuredangle(XF, XA) = \measuredangle(XC, XE)$ transforms to $\measuredangle(f, a) = \measuredangle(c, e)$, i.e. $\angle QSP = \angle QRP$. From this, we get that $PQRS$ is cyclic and thus $\angle SPQ + \angle QRS = 180°$, i.e. $\measuredangle(a, b) + \measuredangle(c, d) = 180°$, which dualized is $\measuredangle(XA, XB) + \measuredangle(XC, XD) = 180°$, i.e. $\angle AXB + \angle CXD = 180°$. ■

Remark. This method is useful when we want to "move" angles that have the same center "away from each other", or alternatively, to "bring closer" angles that are "far away".

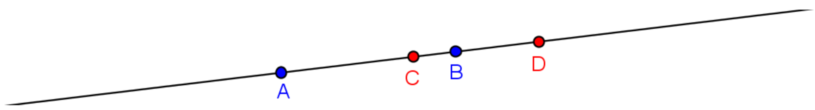
Related problems: (Duality) 90, 121 and 182.

Chapter 22

Harmonic Ratio

If A, B, C and D are collinear points, then their *cross-ratio* is defined as:

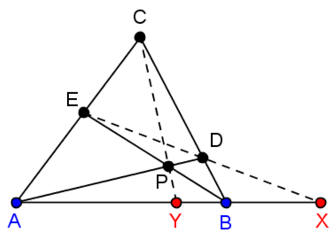
$$(A, B; C, D) = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}.$$



If $(A, B; C, D) = 1$ and the order of the points on the line is such that the line segments AB and CD partially overlap (e.g. $A - C - B - D$), then the ratio is called *harmonic ratio* and the four-tuple $(ACBD)$ is called a *harmonic division*, or simply *harmonic*. The points C and D are *harmonic conjugates* with respect to the points A and B and vice versa. Notice, by the definition, that if $(ACBD)$ is a harmonic, then $(DBCA)$ is also harmonic.

Check for yourself that (HO_9TO) is a harmonic division, where O_9 is the center of the Nine Point Circle and H, T, O are the orthocenter, centroid and circumcenter of a triangle, respectively.

Property 22.1. Let X be a point on the extension of the side AB in $\triangle ABC$. A line which passes through X meets the sides BC and CA at points D and E, respectively. Let P be the intersection of AD and BE. The line CP meets AB at Y. Then, X and Y are harmonic conjugates with respect to the points A and B.



Proof. We need to prove that $\frac{\overline{AX}}{\overline{BX}} = \frac{\overline{AY}}{\overline{BY}}$. By using Menelaus Theorem for $\triangle ABC$ and the collinear points $D - E - X$ and Ceva's Theorem for $\triangle ABC$

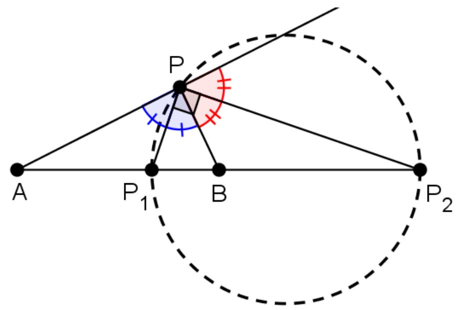
and the concurrent cevians AD, BE and CY, we get:

$$\frac{\overline{AX}}{\overline{XB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1 = \frac{\overline{AY}}{\overline{YB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}.$$

By canceling, we get the needed ratio. ■

Property 22.2. Given two points A and B, find the locus of the points P such that

$$\frac{\overline{AP}}{\overline{PB}} = \lambda,\ \lambda > 0.$$



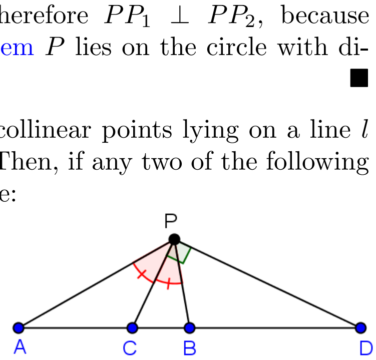
Proof. If $\lambda = 1$, then the locus of the points P is the side bisector of AB. Let's investigate the case when $\lambda \neq 1$. WLOG, let $\lambda > 1$. Obviously, there is a point P_1 between A and B (in this case, closer to B) that satisfies the condition. Also, there is another point, P_2, on the extension of the line (in this case, beyond B), that also satisfies the condition. Note that we know how to construct the latter using Property 22.1. Now let P be a point that doesn't lie on the line AB, but satisfies the condition. Then,

$$\frac{\overline{AP}}{\overline{PB}} = \lambda = \frac{\overline{AP_1}}{\overline{P_1B}} = \frac{\overline{AP_2}}{\overline{P_2B}},$$

so by the Angle Bisector Theorem, we get that PP_1 is the internal angle bisector of $\angle APB$. By the External Angle Bisector Theorem, we get that PP_2 is the external angle bisector of $\angle APB$. Therefore $PP_1 \perp PP_2$, because $\angle P_1PP_2 = \frac{1}{2} \cdot 180° = 90°$, so by Thales' Theorem P lies on the circle with diameter P_1P_2. ■

Property 22.3. Let A, C, B and D be four collinear points lying on a line l in this order. Let P be a point not lying on l. Then, if any two of the following propositions are true, then the third is also true:

1. The division $(ACBD)$ is harmonic.
2. PC is the angle bisector of $\angle APB$.
3. $PC \perp PD$.



Proof. This is a direct consequence of the result of Property 22.2. See its proof for details. ■

Remark. As a consequence to this property, we can see that (AIA_1I_A) is a harmonic division, where I, I_A are the incenter and $A-$excenter of $\triangle ABC$, respectively, and $A_1 = AI \cap BC$.

Property 22.4. Let A, C, B and D be four collinear points lying on a line in this order. Then, the division $(ACBD)$ is harmonic if and only if C is the image of D under inversion with respect to the circle with diameter AB.

Proof. Let O be the midpoint of AB and $r = \frac{1}{2}\overline{AB}$.

$$(ACBD) \text{ is harmonic}$$

$$\iff \frac{\overline{CA}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$$

$$\iff \frac{r + \overline{OC}}{r - \overline{OC}} = \frac{\overline{DO} + r}{\overline{DO} - r}$$

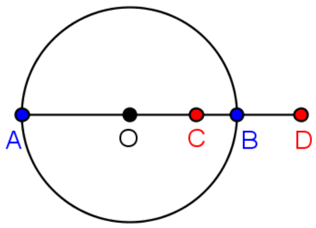
$$\iff (r + \overline{OC}) \cdot (\overline{DO} - r) = (\overline{DO} + r) \cdot (r - \overline{OC})$$

$$\iff r \cdot \overline{OD} - r^2 + \overline{OC} \cdot \overline{OD} - r \cdot \overline{OC} = r \cdot \overline{OD} - \overline{OC} \cdot \overline{OD} + r^2 - r \cdot \overline{OC}$$

$$\iff \overline{OC} \cdot \overline{OD} = r^2$$

$$\iff \mathcal{J}_{O,r} : C \leftrightarrow D$$

■



Property 22.5. Let A, C, B and D be four collinear points lying on a line in this order. Let O be the midpoint of AB. Then, the division $(ACBD)$ is harmonic if and only if $\overline{DA} \cdot \overline{DB} = \overline{DC} \cdot \overline{DO}$ if and only if $\overline{CA} \cdot \overline{CB} = \overline{CD} \cdot \overline{CO}$.

Proof. Let $r = \frac{1}{2}\overline{AB}$.

$$\overline{DA} \cdot \overline{DB} = \overline{DC} \cdot \overline{DO}$$

$$\iff (\overline{OD} + \overline{r}) \cdot (\overline{OD} - \overline{r}) = (\overline{OD} - \overline{OC}) \cdot \overline{OD}$$

$$\iff \overline{OD}^2 - r^2 = \overline{OD}^2 - \overline{OC} \cdot \overline{OD}$$

$$\iff r^2 = \overline{OC} \cdot \overline{OD}$$

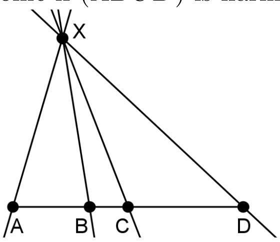
$$\iff \mathcal{J}_{O,r} : C \leftrightarrow D$$

$$\overset{\text{Property 22.4}}{\iff} (ACBD) \text{ is harmonic}$$

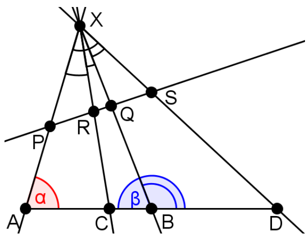
In exactly the same way, we can prove that $\overline{CA} \cdot \overline{CB} = \overline{CD} \cdot \overline{CO}$ iff $(ACBD)$ is harmonic. ■

22.1 Harmonic Pencil

Four points A, B, C and D, are given on a line l in this order. If X is a point not lying on l, then the pencil $X(ABCD)$, which consists of the four lines XA, XB, XC and XD, is harmonic if $(ABCD)$ is harmonic.



Property 22.6. If any pencil $X(ABCD)$ is intersected with another line at points P, Q, R and S, then $(A, B; C, D) = (P, Q; R, S)$. As a consequence, if a harmonic pencil is intersected with a line, the intersection points form a harmonic division.



Proof. WLOG let A, C, B and D (and P, R, Q and S) be collinear in this order. Let $\angle XAC = \alpha$ and $\angle XBC = \beta$. By using the Law of Sines in the triangles $\triangle CXA$, $\triangle CXB$, $\triangle DXA$ and $\triangle DXB$, we get:

$$\frac{\overline{CA}}{\sin(\angle CXA)} = \frac{\overline{CX}}{\sin(\alpha)}$$

$$\frac{\overline{CB}}{\sin(\angle CXB)} = \frac{\overline{CX}}{\sin(\beta)}$$

$$\frac{\overline{DA}}{\sin(\angle DXA)} = \frac{\overline{DX}}{\sin(\alpha)}$$

$$\frac{\overline{DB}}{\sin(\angle DXB)} = \frac{\overline{DX}}{\sin(180^\circ - \beta)}.$$

By rearranging and using that $\sin(\beta) = \sin(180^\circ - \beta)$, we get

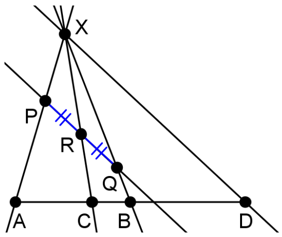
$$(A, B; C, D) = \frac{\sin(\angle CXA)}{\sin(\angle CXB)} : \frac{\sin(\angle DXA)}{\sin(\angle DXB)} \tag{22.1}$$

Since $\angle CXA \equiv \angle RXP$, $\angle CXB \equiv \angle RXQ$, $\angle DXA \equiv \angle SXP$ and $\angle DXB \equiv \angle SXQ$, it follows that $(A, B; C, D) = (P, Q; R, S)$. ■

Remark. Since the cross ratio is not dependend on the line intersecting the pencil, we can define the cross ratio of a pencil $X(ABCD)$ to be

$$(XA, XB; XC, XD) = (A, B; C, D).$$

Property 22.7. Given a pencil $X(ABCD)$ and a line parallel to XD that intersect XA, XB and XC at points P, Q and R, respectively, then $X(ABCD)$ is a harmonic pencil if and only if $\overline{PR} = \overline{RQ}$.



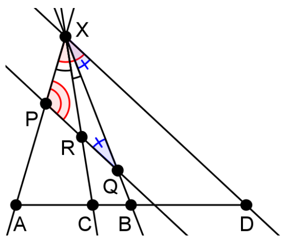
Proof 1. We will firstly give a not so Euclidean proof :)
Since $XD \parallel PQ$, $XD \cap PQ = P_\infty$. Then, by Property 22.6

$$1 = (A, B; C, D) = (P, Q; R, P_\infty) = \frac{\overline{RP}}{\overline{RQ}} : \frac{\overline{PP_\infty}}{\overline{QP_\infty}}$$

Since P_∞ is the point at infinity, then we can take $\overline{PP_\infty} = \overline{QP_\infty}$, giving us $\overline{PR} = \overline{RQ}$. ■

Proof 2. For the more skeptical readers, here is a valid Euclidean proof. Let A, C, B and D be collinear in this order. From Equation 22.1, we know that

$$(A, B; C, D) = \frac{\sin(\angle CXA)}{\sin(\angle CXB)} : \frac{\sin(\angle DXA)}{\sin(\angle DXB)} \tag{1}$$



By using the Law of Sines in the triangles $\triangle PRX$ and $\triangle QRX$, we get

$$\frac{\overline{PR}}{\sin(\angle RXP)} = \frac{\overline{XR}}{\sin(\angle XPR)} \quad \text{and} \quad \frac{\overline{QR}}{\sin(\angle RXQ)} = \frac{\overline{XR}}{\sin(\angle XQR)},$$

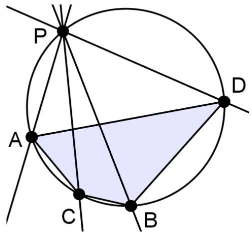
$$\text{i.e. } \frac{\overline{PR}}{\overline{QR}} = 1 \iff \frac{\sin(\angle RXP)}{\sin(\angle RXQ)} : \frac{\sin(\angle XPR)}{\sin(\angle XQR)} = 1. \tag{2}$$

We have $\angle CXA \equiv \angle RXP$ and $\angle CXB \equiv \angle RXQ$. Since $XD \parallel PQ$, we also have $\angle DXA \equiv \angle DXP = 180^\circ - \angle XPR$ and $\angle DXB \equiv \angle DXQ = \angle XQR$. Combining with (1) and (2), we get that

$$\overline{PR} = \overline{QR} \iff (A, B; C, D) = 1$$ ■

22.2 Harmonic Quadrilateral

Let $ABCD$ be a cyclic quadrilateral and P be a point on the circle. Then, $ABCD$ is called *harmonic quadrilateral* if the pencil $P(ABCD)$ is harmonic, i.e. if $(PA, PB; PC, PD) = 1$ and AB and CD intersect inside the circle (the order of the points on the circle is $A - C - B - D - A$, in any direction).



Property 22.8. Let A, C, B and D be points on a circle in this order. Let P be any point on that circle. Then the cross ratio $(PA, PB; PC, PD)$ does not depend on P.

Proof. Let the radius of the circle be R. By the Sine Law for $\triangle CPA$, we get

$$\frac{\overline{CA}}{\sin(\angle CPA)} = 2R.$$

We can get similar equations for the triangles $\triangle CPB$, $\triangle DPA$ and $\triangle DPB$. Therefore, by the defintion of a cross ratio of a pencil and by Equation 22.1, we get

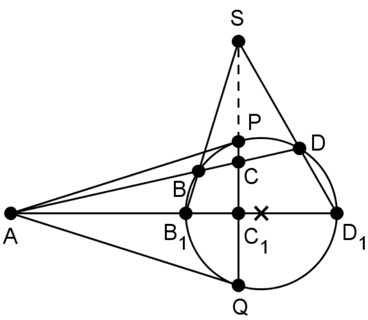
$$(PA, PB; PC, PD) = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)} = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$$

which doesn't depend on the point P.

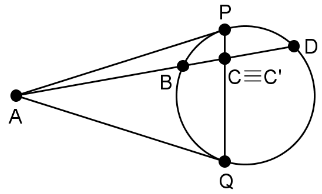
Remark. As a consequence, for a harmonic quadrilateral $ACBD$, the products of the opposite sides are equal, i.e. $\overline{AC} \cdot \overline{BD} = \overline{BC} \cdot \overline{AD}$. ■

Property 22.9. Let A be a point outside of a circle ω. A line l which passes through A, meets ω at points B and D. C is a point on the line segment BD. Prove that the division $(ABCD)$ is harmonic if and only if C lies on the polar of A.

Proof 1. Let P and Q be points on ω such that AP and AQ are tangents. Then, PQ is the polar of A. Let's prove the first direction. Let $C = BD \cap PQ$. We need to prove that $(ABCD)$ is harmonic. Let the secant through A that passes through the center of ω intersect ω at B_1 and D_1 and the line PQ at C_1, such that the points $A - B_1 - C_1 - D_1$ are collinear in that order. Then C_1 is the image of A under inversion with respect to ω. By Property 22.4, we know that $(AB_1C_1D_1)$ is harmonic. Let $B_1B \cap D_1D = S$. By Property 21.3, $S \in a \equiv PQ$. Therefore, B_1B, C_1C and D_1D concur at S. Since $(AB_1C_1D_1)$ is harmonic, then the pencil $S(AB_1C_1D_1)$ is harmonic, too.



When it is intersected by another line, by Property 22.6, the intersection points form a harmonic division, i.e. $(ABCD)$ is harmonic. □



For proving the other direction, let C' be a point on the segment BC such that $(ABC'D)$ is harmonic. We need to prove that $C' \in PQ$. From above, we know that $(ABCD)$ is harmonic, where $C = BD \cap PQ$. Since three of the points in the cross ratio coincide and the cross ratio is equal, then the fourth point must also coincide. We will prove this property once here, but remember it because it is often used in problems.

$$\frac{\overline{BA}}{\overline{BC'}} : \frac{\overline{DA}}{\overline{DC'}} = 1 = \frac{\overline{BA}}{\overline{BC}} : \frac{\overline{DA}}{\overline{DC}}$$

$$\frac{\overline{DC'}}{\overline{BC'}} = \frac{\overline{DC}}{\overline{BC}}$$

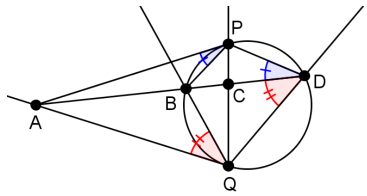
By adding 1 on both sides, we get

$$\frac{\overline{DB}}{\overline{BC'}} = \frac{\overline{DB}}{\overline{BC}}$$

$$\overline{BC'} = \overline{BC}$$

Since we know that both C and C' are between B and D, we get that $C' \equiv C$, i.e. $C' \in PQ$. ■

Proof 2. Again, let P and Q be points on ω such that AP and AQ are tangents and let $C = BD \cap PQ$. We will give an alternate proof of the first direction, i.e. that $(ABCD)$ is harmonic.



It's obvious that $\triangle ABP \sim \triangle APD$ and $\triangle ABQ \sim \triangle AQD$. Since $\overline{AP} = \overline{AQ}$:

$$\frac{\overline{BP}}{\overline{PD}} = \frac{\overline{AB}}{\overline{AP}} = \frac{\overline{AB}}{\overline{AQ}} = \frac{\overline{BQ}}{\overline{QD}}$$

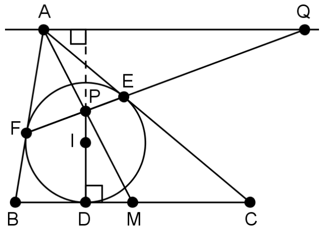
So, $\overline{BP} \cdot \overline{QD} = \overline{PD} \cdot \overline{BQ}$, which by Property 22.8, means that $QBPD$ is a harmonic quadrilateral. Then, the pencil $Q(QBPD)$ is a harmonic pencil. By Property 22.6, we know that if we intersect it by the line AB, then the intersection points $QQ \cap AB = A$, $QB \cap AB = B$, $QP \cap AB = C$ and $QD \cap AB = D$ will form a harmonic division, i.e. $(ABCD)$ is a harmonic.

The other direction is the same as in the previous proof. ■

22.3 Useful Lemmas

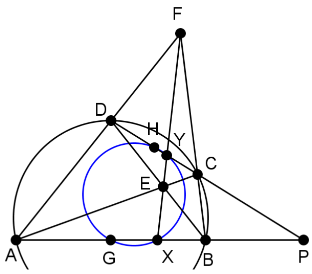
Example 22.1. In $\triangle ABC$, the incircle ω centered at I touches the sides BC, CA and AB at D, E and F, respectively. Let $DI \cap EF = P$ and let $AP \cap BC = M$. Prove that $\overline{BM} = \overline{MC}$.

Proof. We need to prove that $\overline{BM} = \overline{MC}$, so our main idea, by Property 22.7, is to prove that the pencil $A(BMCQ)$ is harmonic, where AQ is some line parallel to BC. Let Q be a point such that $AQ \parallel BC$ and Q lies on the line EF. We will use polars, so let x denote the polar of a point X with respect to ω. AF and AE are tangents to ω, so by Property 21.1, $EF \equiv a$. $P \in a$, so by La Hire's Theorem, $A \in p$. Also, since $IP \perp AQ$ (because $ID \perp BC$ and $BC \parallel AQ$), $AQ \equiv p$. Since $Q \in a$ and $Q \in p$, then by La Hire's Theorem, $AP \equiv q$.



Since $P \in q$, by Property 22.9, the division $(QEPF)$ is harmonic. Then, the pencil $A(QEPF)$ is harmonic. By Property 22.7, by intersecting the harmonic pencil $A(QEPF)$ with the line BC which is parallel to AQ, we get that $(P_\infty CMB)$ is harmonic, i.e. $\overline{CM} = \overline{MB}$. ■

Example 22.2. Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. The line EF intersects AB and CD at X and Y, respectively. Prove that $GXYH$ is a cyclic quadrilateral.



Proof. Let $AB \cap CD = P$. In $\triangle ABF$, the cevians AC, BD and FX are concurrent, so by Property 22.1, we get that $(AXBP)$ is a harmonic. Since $\overline{GA} = \overline{GB}$, by Property 22.5, we get that

$$\overline{PA} \cdot \overline{PB} = \overline{PX} \cdot \overline{PG}. \tag{1}$$

Since $(AXBP)$ is harmonic, then $F(AXBP)$ is a harmonic pencil. If we intersect it with the line CD, by Property 22.6 , the intersection points $(DYCP)$ form a harmonic division. Again, since $\overline{DH} = \overline{HC}$, by Property 22.5, we get that

$$\overline{PC} \cdot \overline{PD} = \overline{PY} \cdot \overline{PH}. \tag{2}$$

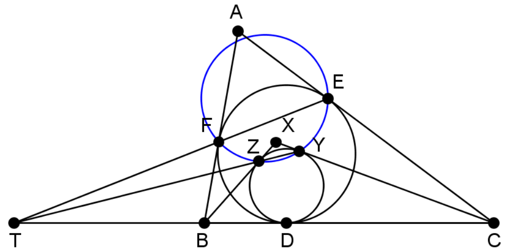
Since $ABCD$ is cyclic, by the Intersecting Secants Theorem, we have

$$\overline{PX} \cdot \overline{PG} \overset{(1)}{=} \overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD} \overset{(2)}{=} \overline{PY} \cdot \overline{PH}.$$

Therefore, by the converse of the Intersecting Secants Theorem, $GXYH$ is a cyclic quadrilateral. ■

Now, let's solve some problems.

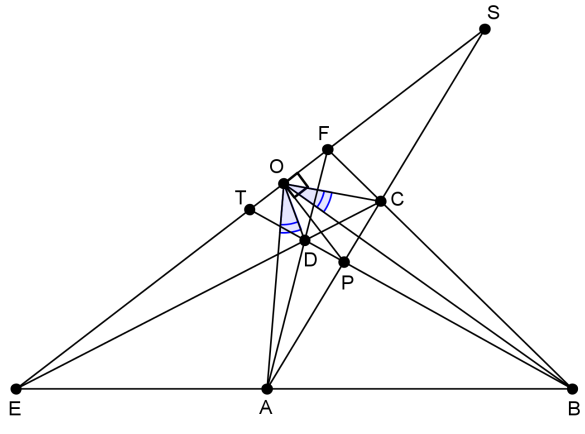
Example 22.3 (IMO Shortlist 1995/G3)**.** The incircle of $\triangle ABC$ touches the sides BC, CA and AB at D, E and F, respectively. X is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches BC at D and touches CX and XB at Y and Z, respectively. Show that E, F, Z and Y are concyclic.



Proof. Let EF intersect BC at T_1. Since AD, BE and CF are concurrent at the Gergonne Point of $\triangle ABC$, by Property 22.1, we get that (T_1BDC) is a harmonic. Similarly, if $YZ \cap BC = T_2$, then (T_2BDC) is a harmonic. Since three of the points are fixed, then the fourth one must also be fixed, i.e. $T_1 \equiv T_2 \equiv T$.

Now, by the Secant-Tangent Theorem for the circle (DEF) and then for the circle (DYZ), we get $\overline{TF} \cdot \overline{TE} = \overline{TD}^2 = \overline{TZ} \cdot \overline{TY}$, which by the converse of the Intersecting Secants Theorem means that E, F, Z and Y are concyclic. ■

Example 22.4 (China TST 2002)**.** Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let O be the foot of the perpendicular from P to EF. Show that $\angle BOC = \angle AOD$.

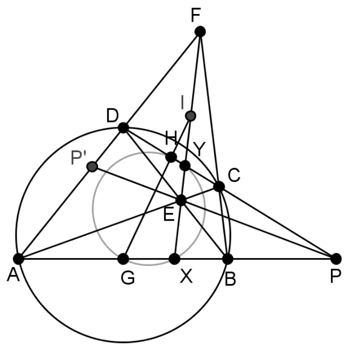


Proof. Let $E = BA \cap CD$ and $F = BC \cap AD$. Also, let $AC \cap EF = S$ and $BD \cap EF = T$. Since the cevians EC, FA and BT in $\triangle EFB$ are concurrent (at D), by Property 22.1, we get that the division $(ETFS)$ is harmonic. Therefore, the pencil $B(ETFS)$ is a harmonic pencil. If we intersect it with the line AC, by Property 22.6, the intersection points also form a harmonic division, i.e. $(APCS)$ is harmonic. Since $OP \perp OS$, by Property 22.3, $\angle AOP = \angle POC$.

On the other hand, since $(APCS)$ is harmonic, the pencil $E(APCS)$ is harmonic, so by intersecting it with the line BD, we get that $(BPDT)$ is harmonic. Again, since $OP \perp OT$, we get $\angle BOP = \angle POD$.

Finally, $\angle BOC = \angle POC - \angle POB = \angle AOP - \angle POD = \angle AOD$. ■

Example 22.5 (IMO Shortlist 2009/G4)**.** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.



Proof 1. Let the line EF intersect the lines AB, CD and GH at X, Y and I, respectively. By Property 19.7, the midpoints of the diagonals of the complete quadrilateral $FDECAB$ are collinear, so I is the midpoint of EF. Let $AB \cap CD = P$ and $PE \cap AD = P'$. In $\triangle ADP$, the cevians AC, DB and PP' are concurrent, so by Property 22.1, $(FDP'A)$ is a harmonic. Therefore, $P(FDP'A)$ is a harmonic pencil. If we intersect it with the line FE, by Property 22.6, the intersection points will form a harmonic division, i.e. $(FYEX)$ is a harmonic. By Property 22.4, $\mathcal{J}_{I,\overline{IE}} : X \leftrightarrow Y$, i.e.

$$\overline{IE}^2 = \overline{IX} \cdot \overline{IY}. \tag{1}$$

From Example 22.2, we know that $GXYH$ is a cyclic quadrilateral, so

$$\overline{IX} \cdot \overline{IY} = \overline{IH} \cdot \overline{IG}. \tag{2}$$

By combining (1) and (2), by the converse of the Secant-Tangent Theorem, we get that $IE \equiv FE$ is tangent to (EHG). ■

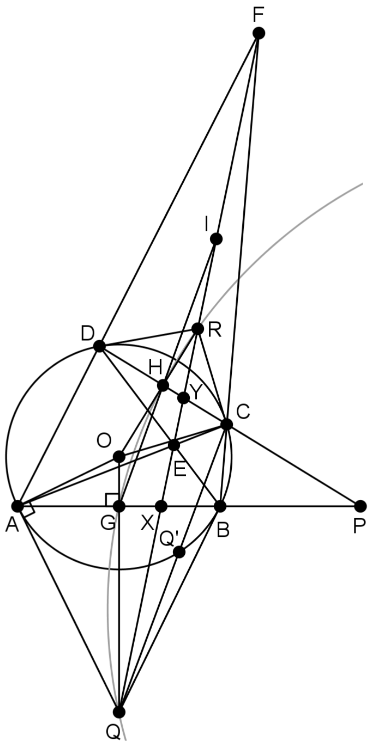
Proof 2. Let $AB \cap CD = P$ and $GH \cap FE = I$. Let $\omega \equiv (ABCD)$ and let O be its center. Let the tangents to ω at A and B intersects at Q. Let the tangents to ω at C and D intersect at R.

Since G is the midpoint of the AB, $G \in OQ$ and $OG \perp AB$. By the Euclid's laws for $\triangle OAQ$, we get $\overline{OA}^2 = \overline{OG} \cdot \overline{OQ}$. Similarly, $\overline{OC}^2 = \overline{OH} \cdot \overline{OR}$. Since $\overline{OA} = \overline{OC}$ as radii in ω, we have $\overline{OG} \cdot \overline{OQ} = \overline{OH} \cdot \overline{OR}$, so by the converse of the Intersecting Secants Theorem $GQRH$ is a cyclic quadrilateral. Therefore, for the secants through the point I, by the Intersecting Secants Theorem, we get

$$\overline{IG} \cdot \overline{IH} = \overline{IQ} \cdot \overline{IR} \tag{1}$$

By Property 21.1, AB is the polar of Q, i.e. $AB \equiv q$. Since $P \in q$, then by La Hire's Theorem, $Q \in p$. On the other hand, from Brocard's Theorem we know that $FE \equiv p$. Therefore, $Q \in FE$. Similarly, $R \in FE$.

Since QA and QB are tangents to (ABC), then by Property 13.3, CQ is a symmedian in $\triangle ABC$. If $Q' = CQ \cap (ABC)$, then by Property 13.5, $AQ'BC$



is a harmonic quadrilateral. Therefore, $C(A, B; Q', C)$ is a harmonic pencil. By intersecting it with the line FE, by Property 22.6, we get that the intersection points form a harmonic division, i.e. $(E, F; Q, R)$ is harmonic. By Property 19.7, the midpoints of the diagonals of the complete quadrilateral $FDECAB$ are collinear, so I is the midpoint of EF. Therefore, by Property 22.4, $\mathcal{J}_{I,\overline{IE}} : Q \leftrightarrow R$, i.e.

$$\overline{IE}^2 = \overline{IQ} \cdot \overline{IR}. \tag{2}$$

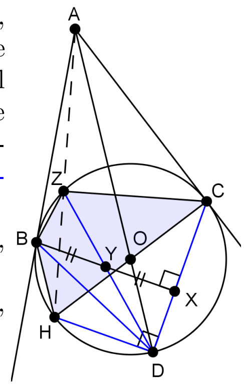
By combining (1) and (2), by the converse of the Secant-Tangent Theorem, we get that $IE \equiv FE$ is tangent to (EHG). ■

Example 22.6 (BMO Shortlist 2007, Cosmin Pohoata)**.** Let ω be a circle centered at O and A be a point outside it. Denote by B and C the points where the tangents from A with respect to ω meet the circle. Let D be the point on ω, for which O lies on the line segment AD. Let X be the foot of the perpendicular from B to CD, Y be the midpoint of the line segment BX and Z be the second intersection of DY with ω. Prove that $ZA \perp ZC$.

Proof. Let $CO \cap \omega = H$. Then, by Thales' Theorem, $DH \perp DC$. Since $XB \perp DC$, we get $DH \parallel XB$. Since $\overline{XY} = \overline{YB}$, by Property 22.7, we get that the pencil (DX, DY, DB, DH) harmonic. Therefore, by definition, the cyclic quadrilateral formed by the intersections of the pencil with ω, $CZBH$, is a harmonic quadrilateral. By Property 13.5, HZ is symmedian in $\triangle HBC$.
Since BA and CA are tangents to ω, then by Property 13.3, HA is a symmedian in $\triangle HBC$.
Finally, $HA \equiv HZ$, i.e. $H - Z - A$ are collinear. Therefore,

$$\angle AZC = 180^\circ - \angle CZH = 180^\circ - 90^\circ = 90^\circ,$$

$$\text{i.e. } ZA \perp ZC$$ ■

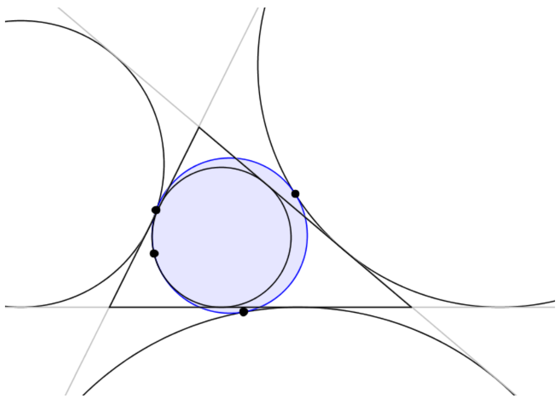


Related problems: 66, 104, 126, 138, 151, 188, 200, 206, 213, 218, 224, 225 and 227.

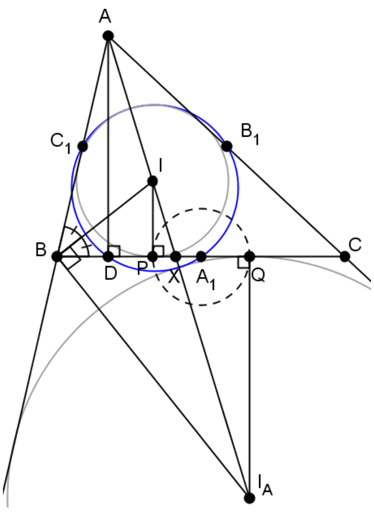
Chapter 23

Feuerbach's Theorem

Property 23.1 (Feuerbach's Theorem)**.** The nine point circle of a triangle is internally tangent to its incircle and externally tangent to its three excircles.



Proof. Let A_1, B_1 and C_1 be the midpoints of BC, CA and AB, respectively. Let ω be the incircle of the triangle, centered at I. Let ω_A be the A-excircle, centered at I_A. Let P and Q be the tangent points of ω and ω_A, respectively to



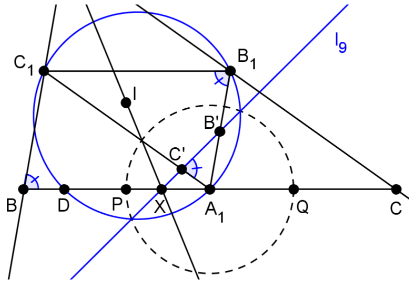
BC. From Property 10.1.3, we know that $\overline{A_1P} = \overline{A_1Q}$. Let $\mathcal{J}$ be the inversion with center A_1 and radius $\overline{A_1P}$. Since A_1P is tangent to ω, $\mathcal{J} : \omega \leftrightarrow \omega$. Similarly $\mathcal{J} : \omega_A \leftrightarrow \omega_A$. Since the nine point circle, C_9, passes through the center of inversion A_1, it will be sent to a line, i.e. $\mathcal{J} : C_9 \leftrightarrow l_9$. Because we want to prove that the nine point circle is tangent to the incircle and A-excircle, we will prove that its image, l_9, is tangent to their images, ω and ω_A.

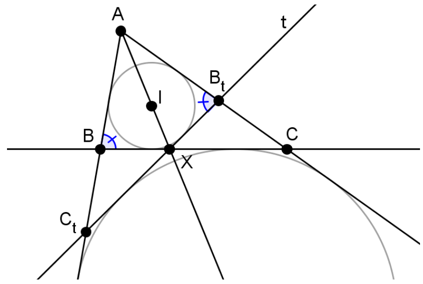
Let $AI \cap BC = X$. Since BI is the internal angle bisector of $\angle ABC$ and $BI \perp BI_A$, by Property 22.3, the division $(AIXI_A)$ is harmonic. Let D be the foot of the altitude from A to BC. Then $AD \parallel IP \parallel I_AQ$ (because they are all perpendicular to BC). So, by Thales'

Proportionality Theorem, the division $(DPXQ)$ is also harmonic. By Property 22.4, since PQ is the diameter of the circle of inversion, $\mathcal{J} : D \leftrightarrow X$. Since $D \in C_9$, then $X \in l_9$. Let $\mathcal{J} : B_1 \leftrightarrow B'$ and $\mathcal{J} : C_1 \leftrightarrow C'$. Then by Property 20.1

$$\angle A_1C'B' = \angle A_1B_1C_1 = \angle ABC. \tag{1}$$

Also, since $B_1, C_1 \in C_9$, then $B', C' \in l_9$.





Since X is the intersection of the line connecting the centers of ω and ω_A, II_A, and one of the common internal tangents, BC, then the other common internal tangent, t, must also pass through X. We want to prove that $l_9 \equiv t$. Let $t \cap AB = C_t$ and $t \cap AC = B_t$. By symmetry with respect to the line AI,

$$\angle AB_tC_t = \angle ABC. \tag{2}$$

From (1), we know that $\angle(A_1C_1, l_9) = \angle ABC$. From (2), we know that $\angle(AC, t) = \angle ABC$. Since $A_1C_1 \parallel AC$, it means that $l_9 \parallel t$. But we already know that $X \in l_9$ and $X \in t$, so $l_9 \equiv t$.

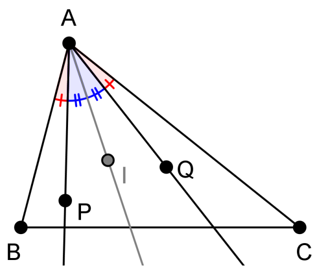
Therefore, the nine point circle is tangent to the incircle and the A-excircle. Similarly, it is tangent to the other two excircles. ■

Remark. The tangent point of the incircle and the nine point circle is called the *Feuerbach point* of the triangle.

Chapter 24

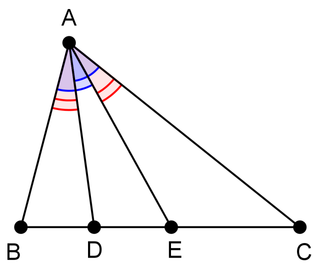
Isogonal Lines

The lines AP, AQ are *isogonal* with respect to the lines AB, AC (or with respect to $\angle BAC$) if and only if $\angle PAB = \angle QAC$, i.e. AP and AQ are reflections with respect to the angle bisector of $\angle BAC$.



Property 24.1 (Steiner's Ratio Theorem)**.** Let $D, E \in BC$, such that AD, AE are isogonal with respect to $\angle BAC$. Then

$$\frac{\overline{BD}}{\overline{CD}} \cdot \frac{\overline{BE}}{\overline{CE}} = \left(\frac{\overline{AB}}{\overline{AC}}\right)^2.$$



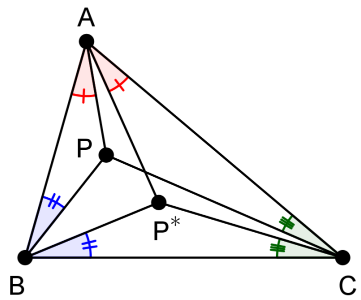
Proof.

$$\frac{\overline{BD}}{\overline{CD}} = \frac{P_{\triangle BDA}}{P_{\triangle CDA}} = \frac{\overline{AB} \cdot \overline{AD} \cdot \sin \angle BAD}{\overline{AC} \cdot \overline{AD} \cdot \sin \angle CAD}$$
$$\frac{\overline{BE}}{\overline{CE}} = \frac{P_{\triangle BEA}}{P_{\triangle CEA}} = \frac{\overline{AB} \cdot \overline{AE} \cdot \sin \angle BAE}{\overline{AC} \cdot \overline{AE} \cdot \sin \angle CAE}$$

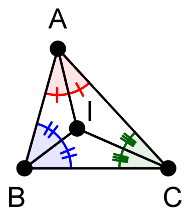
By multiplying these two equalities and using $\angle BAD = \angle CAE$ and $\angle BAE = \angle CAD$, we get the desired result. ∎

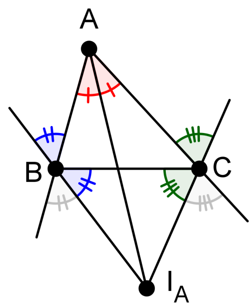
24.1 Isogonal Conjugates

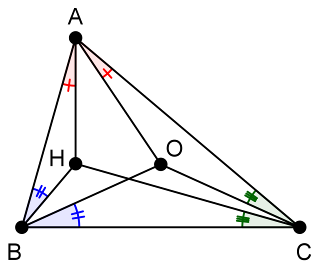
The *isogonal conjugate* of the point P with respect to $\triangle ABC$ is the point P^* such that the lines XP, XP^* are isogonal with respect to every angle $\angle X$ of $\triangle ABC$. If P^* is the isogonal conjugate of P with respect to $\triangle ABC$, then P is the isogonal conjugate of P^*.

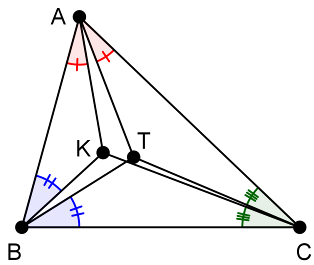


The isogonal conjugate of the incenter I is itself. Also, the isogonal conjugate of an excenter e.g. I_A is itself. By Property 6.9, the orthocenter H and the circumcenter O are isogonal conjugates. By definition, the centroid T and the Lemoine Point K are isogonal conjugates.







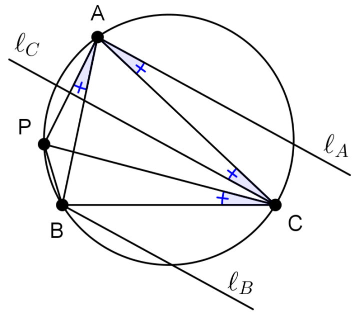


Property 24.1.1. Let P be a point in the plane of $\triangle ABC$. Let ℓ_A be the line isogonal to AP w.r.t. $\angle BAC$. Similarly define lines ℓ_B and ℓ_C. Prove that ℓ_A, ℓ_B and ℓ_C are parallel if and only if $P \in (ABC)$.

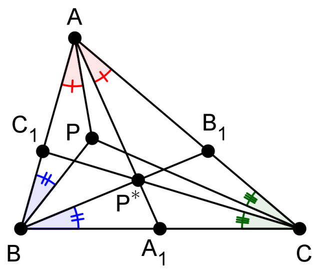
Proof.

$$\begin{aligned} P \in (ABC) &\iff \angle PAB = \angle PCB \\ &\iff \measuredangle(AP, AB) = \measuredangle(CP, CB) \\ &\iff \measuredangle(\ell_A, AC) = \measuredangle(\ell_C, CA) \\ &\iff \ell_A \parallel \ell_C \end{aligned}$$

Similarly, $P \in (ABC) \iff \ell_B \parallel \ell_C$. ■



Property 24.1.2. For any point P that doesn't lie on the sidelines of $\triangle ABC$, there exists a unique point P^*, such that P^* is isogonal conjugate of P.



Proof. Since the lines AP, BP and CP are concurrent at P, by Trigonometric Ceva's Theorem we get

$$\frac{\sin \angle BAP}{\sin \angle CAP} \cdot \frac{\sin \angle CBP}{\sin \angle ABP} \cdot \frac{\sin \angle ACP}{\sin \angle BCP} = 1.$$

Let $A_1 \in BC$, such that AP, AA_1 are isogonal w.r.t. $\angle A$. Therefore, $\angle BAP = \angle CAA_1$ and $\angle CAP = \angle BAA_1$. Similarly, we define points B_1 and C_1. If we substitute the 6 pairs of equal angles in the above equation, we get

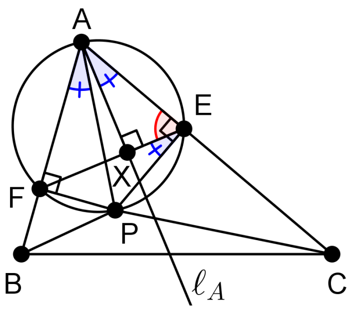
$$\frac{\sin \angle CAA_1}{\sin \angle BAA_1} \cdot \frac{\sin \angle ABB_1}{\sin \angle CBB_1} \cdot \frac{\sin \angle BCC_1}{\sin \angle ACC_1} = 1,$$

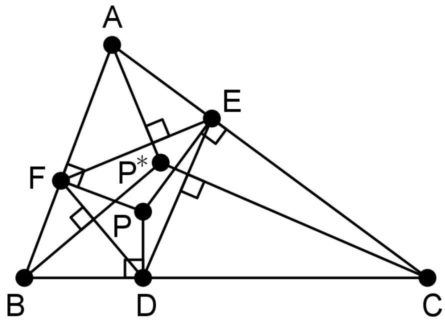
so by the converse of Trigonometric Ceva's Theorem, we get that lines AA_1, BB_1 and CC_1 are concurrent or all parallel.

i) If they are concurrent, then the point of concurrency P^* is an isogonal conjugate of P. The lines AA_1 and BB_1 can't intersect in more than one point, so P^* is the unique point that is an isogonal conjugate of P.

ii) If they are parallel, then we consider the point at infinity P_∞ as the isogonal conjugate of P. From Property 24.1.1, we know that this happens only if $P \in (ABC)$. ■

Property 24.1.3. Let P be a point inside $\triangle ABC$. Let $\triangle DEF$ be the pedal triangle[1] of P. Let ℓ_A be the line through A perpendicular to EF. Similarly define lines ℓ_B and ℓ_C. Then, the lines ℓ_A, ℓ_B and ℓ_C are concurrent at P^*.

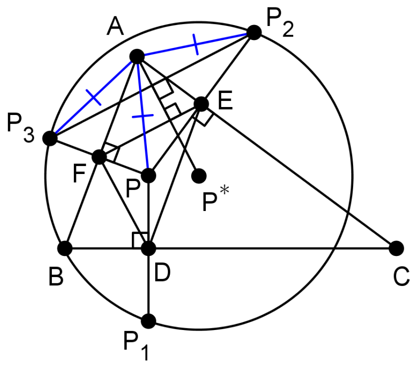




Proof 1. Since $\angle PEA + \angle PFA = 180°$, we get $AFPE$ is cyclic and therefore $\varphi = \angle FAP = \angle FEP$. Then, $\angle FEA = 90° - \varphi$. Let $\ell_A \cap EF = X$. Now, from $\triangle AEX$, $\angle XAE = \varphi$, so $\measuredangle(AP, AB) = \measuredangle(\ell_A, AC)$, i.e. ℓ_A and AP are isogonal w.r.t. $\angle A$. By Property 24.1.2, $P^* \in \ell_A$. Similarly, $P^* \in \ell_B$ and $P^* \in \ell_C$. ■

Proof 2. This concurrence can also be proven by Property 9.2 and then we can do the same angle chase that we did in Proof 1 in order to show that the lines are isogonal. ■

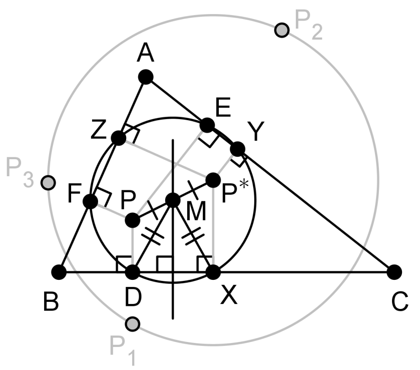
Property 24.1.4. Let P be a point inside $\triangle ABC$. Let P_1, P_2, P_3 be the reflections of P with respect to the sides BC, CA, AB, respectively. Prove that P^* is the circumcenter of $\triangle P_1P_2P_3$.



Proof. Let D, E, F be the corresponding feet of the perpendiculars from P to the sides of the triangle. Then, EF is midsegment in $\triangle PP_2P_3$, so $EF \parallel P_2P_3$. From Property 24.1.3, we know that $AP^* \perp EF$, so $AP^* \perp P_2P_3$. Also, from reflection properties, we get $\overline{AP_2} = \overline{AP} = \overline{AP_3}$. So, $\triangle AP_2P_3$ is isosceles and $AP^* \perp P_2P_3$, thus AP^* is the side bisector of P_2P_3. Similarly, P^* lies on the side bisectors of P_3P_1 and P_1P_2, so P^* is the circumcenter of $\triangle P_1P_2P_3$. ■

[1]The pedal triangle of a point P with respect to $\triangle ABC$ is the triangle with vertices the feet of the perpendiculars from P to the sides of the triangle.

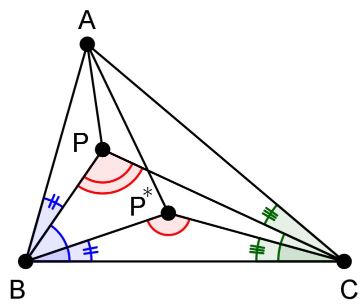
Property 24.1.5. Let P and P^* be a pair of isogonal conjugates in $\triangle ABC$. Let $\triangle DEF$ and $\triangle XYZ$ be the pedal triangles of P and P^*, respectively. Then, the points D, E, F, X, Y, Z are concyclic and the center of this circle is the midpoint of PP^*.



Proof. Let P_1, P_2, P_3 be the reflections of P with respect to the sides BC, CA, AB, respectively. From the definitions of projection and reflection, we get that $\triangle DEF$ and $\triangle P_1P_2P_3$ are homothetic with center P and ratio 2. From Property 24.1.4 we know that P^* is the circumcenter of $P_1P_2P_3$. Therefore, the center of (DEF) is $\frac{1}{2}$-way from P to P^*, i.e. it's the midpoint of PP^*, M. Similarly, the center of (XYZ) is the midpoint of P^*P, which is again M. Finally, M lies on the midsegment in the right trapezoid DXP^*P, so M lies on the side bisector of DX. Therefore, $\overline{MD} = \overline{MX}$, i.e. (DEF) and (XYZ) have the same center M and equal radii, so they must be the same circle. ■

Property 24.1.6. Let P be a point inside $\triangle ABC$ and let P^* be its isogonal conjugate. Then,

$$\angle BPC + \angle BP^*C = 180^\circ + \angle BAC.$$



Proof.

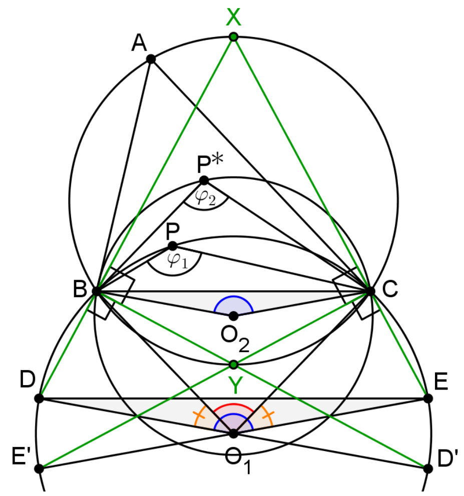
$$\angle BPC = 180^\circ - \angle PBC - \angle PCB$$

$$\angle BP^*C = 180^\circ - \angle P^*BC - \angle P^*CB = 180^\circ - \angle PBA - \angle PCA$$

$$\therefore \angle BPC + \angle BP^*C = 360^\circ - \angle ABC - \angle BCA = 180^\circ + \angle BAC$$ ■

Property 24.1.7. Let P be a point inside $\triangle ABC$ and let P^* be its isogonal conjugate. Then, the exsimilicenter and the insimilicenter of (BPC) and (BP^*C) lie on (ABC).

Proof. From section 17.1 we know that if we have two circles ω_1 and ω_2 with parallel diameters A_1B_1 and A_2B_2, respectively, then $A_1A_2 \cap B_1B_2$ is their exsimilicenter. Let O_1, O_2 be the centers of $(BPC), (BP^*C)$, respectively. So, if $D \in (BPC)$ such that $BO_2 \parallel DO_1$, then the exsimilicenter lies on BD. Similarly, if $E \in (BPC)$ such that $CO_2 \parallel EO_1$, it lies on CE, so $BD \cap CE = X$ is the exsimilicenter of (BPC) and (BP^*C). Triangles $\triangle BO_2C$ and $\triangle DO_1E$ are similar with two pairs of parallel sides, so we must have $BC \parallel DE$ and thus $\widehat{BD} = \widehat{CE}$, i.e. $\angle BO_1D = \angle CO_1E$. Let $\varphi_1 = \angle BPC$ and $\varphi_2 = \angle BP^*C$.



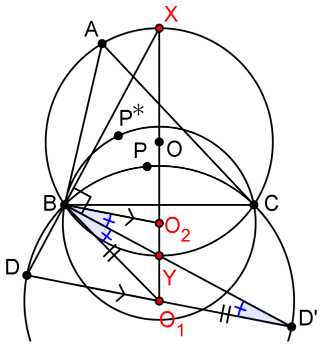
$$\angle XDE \equiv \angle BDE = \frac{1}{2}\angle BO_1E = \frac{1}{2}(\angle BO_1C + \angle CO_1E) =$$
$$= \frac{1}{2}\left(\angle BO_1C + \frac{\angle DO_1E - \angle BO_1C}{2}\right) = \frac{1}{2}\left(\frac{\angle BO_1C}{2} + \frac{\angle DO_1E}{2}\right) =$$
$$= \frac{1}{2}\left(\frac{\angle BO_1C}{2} + \frac{\angle BO_2C}{2}\right) = \frac{1}{2}(180° - \varphi_1 + 180° - \varphi_2) =$$
$$= 180° - \frac{\varphi_1 + \varphi_2}{2}$$

Since $X \in O_1O_2$, $O_1O_2 \perp BC$ and $BC \parallel DE$, we have $\triangle XDE$ is isosceles, so $\angle DXE = 180° - 2\angle XDE = 180° - 360° + \varphi_1 + \varphi_2 = \varphi_1 + \varphi_2 - 180°$.

From Property 24.1.6, we know that $\varphi_1 + \varphi_2 = 180° + \angle BAC$, so $\angle BXC \equiv \angle DXE = \angle BAC$ and thus $X \in (ABC)$. □

From section 17.1 we know that if we have two circles ω_1 and ω_2 with parallel diameters A_1B_1 and A_2B_2, respectively, then $A_1B_2 \cap B_1A_2$ is their insimilicenter. Let $D' \in (BPC)$, such that DD' is diameter. Since $DO_1 \parallel BO_2$, the insimilicenter lies on BD'. Similarly, let $E' \in (BPC)$, such that EE' is diameter. Since $EO_1 \parallel CO_2$, it lies on CE', so $BD' \cap CE' = Y$ is the insimilicenter of (BPC) and (BP^*C). Since DD' is diameter, we have $\angle DBD' = 90°$. Thus, $\angle XBY \equiv \angle XBD' = 180° - \angle DBD' = 90°$. Similarly, $\angle XCY = 90°$, so $XBYC$ is cyclic, i.e. $Y \in (XBC) \equiv (ABC)$. ■

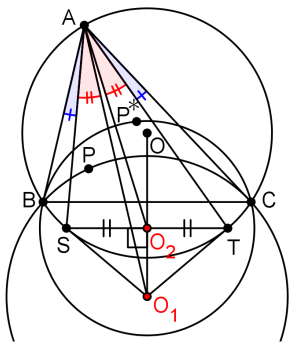
Property 24.1.8. Let P be a point inside $\triangle ABC$ and let P^* be its isogonal conjugate. Let O, O_1 and O_2 be the centers of (ABC), (BPC) and (BP^*C), respectively. Prove that $\mathcal{J}_{O,\overline{OA}} : O_1 \leftrightarrow O_2$.



Proof. Using the notations from the previous property, since $X, Y \in O_1O_2$ (as homothetic centers) and O, O_1, O_2 are collinear (as centers of coaxial circles), we get that O_1, O_2 lie on the line of the diameter XY of (ABC), so by Property 22.4, it is enough to prove that $(O_1, O_2; X, Y)$ is harmonic.

In the proof of the previous property, we had $BO_2 \parallel DO_1$ and since $\overline{O_1B} = \overline{O_1D'}$, we get $\angle O_1BY \equiv \angle O_1BD' = \angle O_1D'B = \angle D'BO_2 \equiv \angle YBO_2$, i.e. BY is angle bisector of $\angle O_1BO_2$. Also, we already showed that $BX \perp BY$. Therefore, by Property 22.3, we get that $(O_1, O_2; X, Y)$ is harmonic. ■

Property 24.1.9. Let P be a point inside $\triangle ABC$ and let P^* be its isogonal conjugate. Let O_1 and O_2 be the centers of (BPC) and (BP^*C), respectively. Prove that AO_1, AO_2 are isogonal with respect to $\angle BAC$.



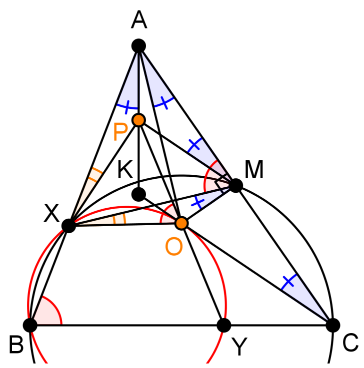
Proof. Let $S, T \in (ABC)$, such that $ST \ni O_2$ and $ST \parallel BC$. Then, $\angle BAS = \angle CAT$ and $\overline{SO_2} = \overline{O_2T}$. Since $OO_1 \perp ST$ and $\mathcal{J}_{O,\overline{OA}} : O_1 \leftrightarrow O_2$ (by Property 24.1.8), by definition of polar, ST is polar of O_1 w.r.t. (ABC). Therefore, by Property 21.1, O_1S, O_1T are tangents to (ABC). Now, by Property 13.3, AO_1 is symmedian in $\triangle AST$. Since AO_2 is median in $\triangle AST$, by definition of symmedian, we get $\angle SAO_1 = \angle TAO_2$. Thus, $\angle BAO_1 = \angle CAO_2$. ■

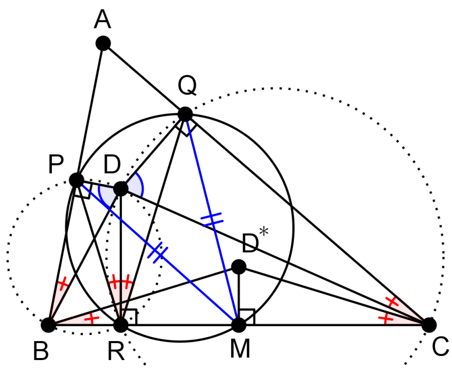
Now, let's solve a few Olympiad problems.

Example 24.1.1 (IGO 2017, Advanced)**.** Let O be the circumcenter of triangle ABC. Line CO intersects the altitude from A at point K. Let P, M be the midpoints of AK, AC respectively. If PO intersects BC at Y, and the circumcircle of triangle BCM meets AB at X, prove that $BXOY$ is cyclic.

Proof. Since AK is an altitude in $\triangle ABC$ and the circumcenter and orthocenter are isogonal conjugates (Property 6.9), we get $\angle OAC = \angle KAB = 90° - \beta$. Since $\overline{OA} = \overline{OC}$ as radii and since PM is a midsegment in $\triangle AKC$, we get $\angle PMA = \angle KCA \equiv \angle OCA = \angle OAC = 90° - \beta$. Since $BCMX$ is cyclic, we have $\angle AMX = \beta$ and since $\angle OMA = 90°$, we get $\angle OMX = 90° - \beta = \angle PMA$. Combining with $\angle OAM = 90° - \beta = \angle PAX$, we conclude that O and P are isogonal conjugates in $\triangle MAX$. Therefore, $\angle OXM = \angle PXA$.

Now, by AA, we get that $\triangle XMO \sim \triangle XAP$. Therefore, by SAS (or by Property 18.3), we get $\triangle XMA \sim \triangle XOP$. Therefore, $\angle XOP = \angle XMA = \beta$. Combining with $\angle XBY = \beta$, we get that $BXOY$ is cyclic. ∎





Example 24.1.2. Let P and Q be the projections of a point D inside $\triangle ABC$ to its sides AB and AC, respectively. Let M be the midpoint of side BC. If $\angle BDP = \angle CDQ$, prove that $\overline{MP} = \overline{MQ}$.

Proof. By the given condition, the right triangles $\triangle DPB$ and $\triangle DQC$ are similar, so $\angle DBP = \angle DCQ$, i.e. $\angle DBA = \angle DCA$. Let D^* be the isogonal conjugate of D w.r.t. $\triangle ABC$. Then, $\angle D^*BC = \angle DBA = \angle DCA = \angle D^*CB$. Therefore, $\triangle D^*BC$ is isosceles, so $D^*M \perp BC$. Let R be the projection of D onto BC. By Property 24.1.5, the points P, Q, R, M are concyclic.

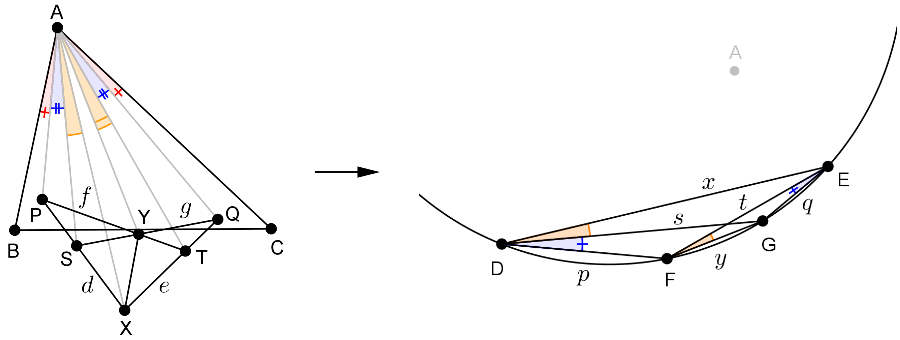
Since $\angle DPB + \angle DRB = 180°$, we get $DPBR$ is cyclic. Similarly, $DRCQ$ is cyclic and therefore $\angle DRP = \angle DBP = \angle DCQ = \angle DRQ$, i.e. DR is internal angle bisector of $\angle PRQ$. Since $RD \perp RM$, we get that RM is external angle bisector of $\angle PRQ$. Since $M \in (PQR)$, we get $\widehat{MP} = \widehat{MQ}$, i.e. $\overline{MP} = \overline{MQ}$. ∎

Related problems: 105, 168 and 192.

24.2 Isogonal Lines Lemma

Property 24.2.1 (Isogonal Lines Lemma)**.** Let AP, AQ and AS, AT be two pairs of isogonal lines with respect to $\angle BAC$. Let $PS \cap QT = X$ and $PT \cap QS = Y$. Then, AX, AY are isogonal lines with respect to $\angle BAC$.

Proof 1. This proof is due to [2]. We will use Pole-Polar Duality with center A and arbitrary radius. We will use lowercase letters to denote the polars of the points in uppercase (e.g. p is the polar of the point P).



Let $d \equiv PS$, $e \equiv QT$, $f \equiv PT$ and $g \equiv QS$. Since $d \equiv PS$, by La Hire's Theorem, we get that $D = p \cap s$. Similarly, $E = q \cap t$, $F = p \cap t$ and $G = q \cap s$. Also, since $X = d \cap e$, we get that $x \equiv DE$ and similarly $y \equiv FG$.

Since $\measuredangle(AP, AS) = \measuredangle(AT, AQ)$, by Property 21.5, we get that $\measuredangle(p, s) = \measuredangle(t, q)$, i.e. $\angle FDG = \angle FEG$. Therefore, $FDEG$ is cyclic and thus $\angle GDE = \angle GFE$, i.e. $\measuredangle(s, x) = \measuredangle(y, t)$, which again by Property 21.5 is $\measuredangle(AS, AX) = \measuredangle(AY, AT)$. Therefore, $\measuredangle(AB, AX) = \measuredangle(AY, AC)$. ∎

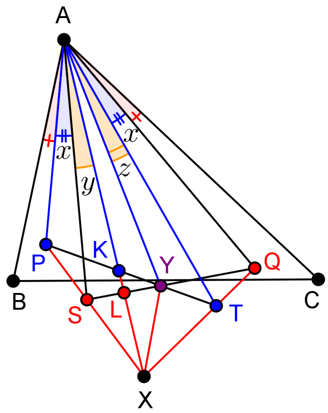
Proof 2. Let $AX \cap PT = K$ and $AX \cap QS = L$. Then,

$$(P, Y; K, T) = (XP, XY; XK, XT) \equiv (XS, XY; XL, XQ) = (S, Y; L, Q)$$

$$\therefore \quad \frac{\sin(\angle KAP)}{\sin(\angle KAY)} : \frac{\sin(\angle TAP)}{\sin(\angle TAY)} = \frac{\sin(\angle LAS)}{\sin(\angle LAY)} : \frac{\sin(\angle QAS)}{\sin(\angle QAY)}$$

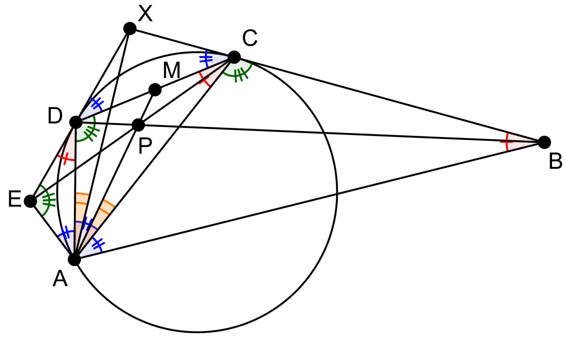
Let $x = \angle PAS = \angle QAT$, $y = \angle SAX$ and $z = \angle TAY$. Since $\angle KAY \equiv \angle LAY$ and $\angle TAP = \angle QAS$, we get

$$\sin(x+y) \cdot \sin z = \sin y \cdot \sin(x+z)$$
$$(\sin x \cos y + \cos x \sin y) \cdot \sin z =$$
$$= \sin y \cdot (\sin x \cos z + \cos x \sin z)$$
$$\sin x \cos y \sin z = \sin y \sin x \cos z$$
$$\tan z = \tan y$$



Since $0° < y, z < 180°$, we get $y = z$, and therefore $\angle XAB = \angle YAC$. ∎

Example 24.2.1 (IMO Shortlist 2006)**.** Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle CBA = \angle DCA = \angle EDA$. Diagonals BD and CE meet at P. Prove that AP bisects side CD.

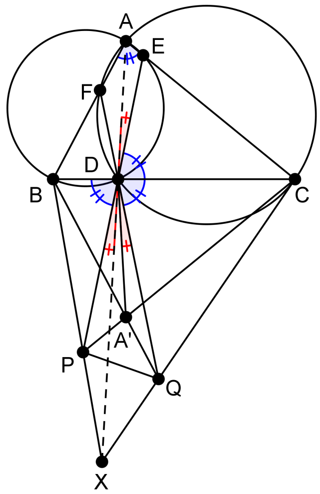


Proof. We see that AC, AD and AB, AE are two pairs of isogonal lines with respect to $\angle CAD$. We have $CB \cap DE = X$ and $CE \cap DB = P$, so by Isogonal Lines Lemma we get that AX, AP are isogonal lines with respect to $\angle CAD$.

$$\angle CDX = 180^\circ - (\angle ADC + \angle EDA) = 180^\circ - (\angle ADC + \angle DCA) = \angle CAD$$

Therefore, DX is tangent to (ACD). Similarly, CX is tangent to (ACD). By Property 13.3, AX is symmedian in $\triangle ACD$. Therefore, AP is the line that is isogonal to the symmedian, i.e. AP is median in $\triangle ACD$. ■

Example 24.2.2 (RMM 2016)**.** Let ABC be a triangle and let D be a point on the segment BC, $D \neq B$ and $D \neq C$. The circle (ABD) meets the segment AC again at an interior point E. The circle (ACD) meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).



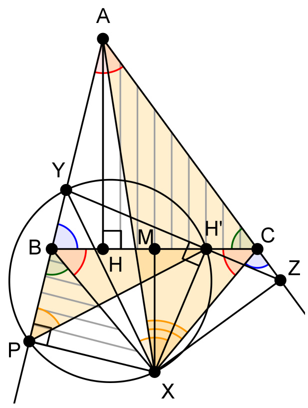
Proof. Since $ABDE$ and $ACDF$ are cyclic, we have

$$\angle BDP = \angle BAE \equiv \angle FAC = \angle QDC.$$

Therefore, DB, DC and DP, DQ are two pairs of isogonal lines w.r.t. $\angle PDQ$. By Isogonal Lines Lemma, if $BP \cap CQ = X$ and $BQ \cap CP = A'$, we get that DX and DA' are isogonal w.r.t. $\angle PDQ$. Therefore, $\angle PDX = \angle QDA'$. Also, since $\angle EDC = \angle PDB = \angle QDC$, we get that the lines DQ and DE are reflections of each other w.r.t. BC. Therefore, $\angle QDA' = \angle EDA$. Finally, we get that $\angle PDX = \angle EDA$ and since $P - D - E$ are collinear, then so must be $X - D - A$. ■

Example 24.2.3 (Iran TST 2015)**.** Let H be the foot of the $A-$altitude in $\triangle ABC$ and H' be the reflection of H through the midpoint of BC. The tangent lines to the circumcircle of $\triangle ABC$ at B and C intersect at X. If the perpendicular line to XH' at H' intersects AB and AC at Y and Z, respectively, prove that $\angle ZXC = \angle YXB$.

Proof. We need to prove that XB, XZ are isogonal lines w.r.t. $\angle YXC$. Obviously, XY, XC is one pair of isogonal lines w.r.t. $\angle YXC$.



Since $B = YA \cap CH'$ and $Z = YH' \cap AC$, we need to prove that XA, XH' is the second pair of isogonal lines w.r.t. $\angle YXC$, i.e. we need to prove that $\angle YXH' = \angle CXA$.

Let P be the foot of the perpendicular from X to AB. Then, $XPYH'$ is cyclic and therefore $\angle YXH' = \angle YPH' \equiv \angle BPH'$. But, $\angle PBH' = 180° - \beta = \alpha + \gamma = \angle XCA$, so we need to prove that $\triangle PBH' \sim \triangle XCA$. For that, by SAS, we need to prove that

$$\frac{\overline{PB}}{\overline{XC}} = \frac{\overline{BH'}}{\overline{CA}} \iff \frac{\overline{PB}}{\overline{BX}} = \frac{\overline{HC}}{\overline{CA}}.$$

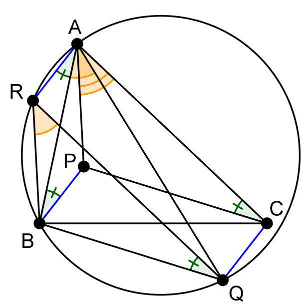
This is true because we have $\triangle PBX \sim \triangle HCA$ (by AA). ■

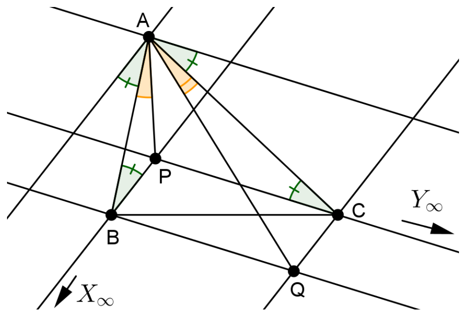
Related problems: 96, 140, 152, 156, 184 and 214.

24.3 Parallelogram Isogonality Lemma

Property 24.3.1 (Parallelogram Isogonality Lemma)**.** (British MO 2, 2013) The point P lies inside triangle ABC so that $\angle ABP = \angle PCA$. The point Q is such that $PBQC$ is a parallelogram. Prove that $\angle QAB = \angle CAP$.

Proof 1. Let R be a point such that $BPAR$ is a parallelogram. Then, $\overline{AR} = \overline{PB} = \overline{CQ}$ and $AR \parallel PB \parallel CQ$. Therefore, $\triangle RBQ$ is the image of $\triangle APC$ with respect to a translation with vector $\overrightarrow{AR}$, so $\triangle RBQ \cong \triangle APC$. Thus, $\angle RQB = \angle ACP = \angle PBA = \angle RAB$, so $RAQB$ is cyclic. Finally, $\angle BAQ = \angle BRQ = \angle PAC$. ■





Proof 2. Let X_∞ be the point at infinity on lines $BP \parallel CQ$ and let Y_∞ be the point at infinity on lines $CP \parallel BQ$. Then, $AX_\infty \parallel BP$ and $AY_\infty \parallel CP$, so

$$\measuredangle(AX_\infty, AB) = \angle ABP = \angle ACP = \measuredangle(AY_\infty, AC).$$

Therefore, AX_∞, AY_∞ and AB, AC are two pairs of isogonal lines w.r.t. $\angle BAC$, so by using Isogonal Lines Lemma and observing that $BX_\infty \cap CY_\infty = P$ and $BY_\infty \cap CX_\infty = Q$, we get that AP, AQ are isogonal w.r.t. $\angle BAC$. ■

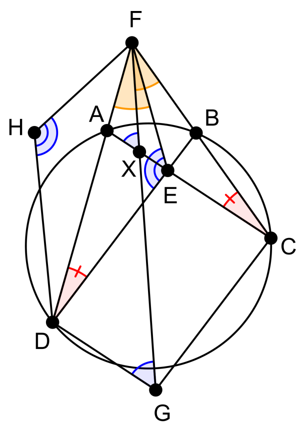
Remark. Notice from the proofs above that the converse is also true, i.e. if $\angle QAB = \angle CAP$, then $\angle ABP = \angle PCA$.

Example 24.3.1 (IMO Shortlist 2012/G2)**.** Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F and G are concyclic.

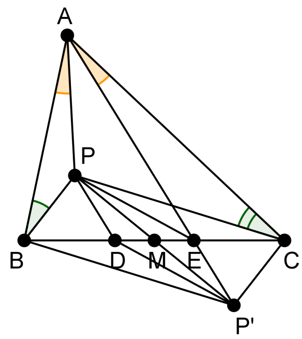
Proof. From $\angle FDE \equiv \angle ADB = \angle ACB \equiv \angle ECF$ and since $EDGC$ is a parallelogram, by Parallelogram Isogonality Lemma, we get that the lines FE, FG are isogonal w.r.t. $\angle DFC$. Let $AC \cap FG = X$.

$$\begin{aligned}\angle DGF = \angle AXF = \angle XCF + \angle XFC \equiv\\ \equiv \angle ACB + \angle GFC = \angle ADB + \angle EFD \equiv\\ \equiv \angle FDE + \angle EFD = 180^\circ - \angle DEF =\\ = 180^\circ - \angle DHF,\end{aligned}$$

so $DGFH$ is cyclic. ■



Example 24.3.2 (ELMO 2012)**.** Let ABC be an acute triangle with $\overline{AB} < \overline{AC}$ and let D and E be points on side BC such that $\overline{BD} = \overline{CE}$ and D lies between B and E. Suppose there exists a point P inside $\triangle ABC$ such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.



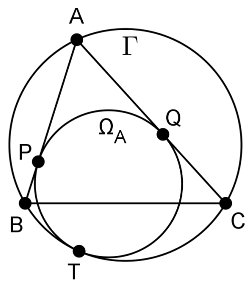
Proof. Let M be the midpoint of BC and let P' be the reflection of P w.r.t. M. Then, $\overline{DM} = \overline{ME}$ and $\overline{PM} = \overline{MP'}$, so, by Property 2.18, $PDP'E$ is a parallelogram. Therefore, $EP' \parallel PD \parallel AE$, so $A - E - P'$ are collinear. Therefore, AP, AP' are isogonal w.r.t. $\angle BAC$. Similarly, $BPCP'$ is also a parallelogram, so by the converse of Parallelogram Isogonality Lemma, we get that $\angle PBA = \angle PCA$. ■

Related problems: 100 and 198.

Chapter 25

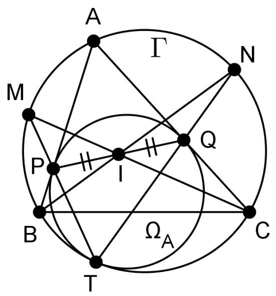
Mixtilinear Incircles

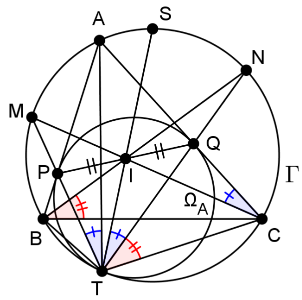
A circle that is internally tangent to two sides of a triangle and to its circumcircle is called a *mixtilinear incircle*. In a triangle, there are three mixtilinear incircles, one corresponding to each angle of the triangle.
Let Ω_A be the $A-$mixtilinear incircle in $\triangle ABC$ and let its tangent points to AB, AC and $\Gamma \equiv (ABC)$ be P, Q and T, respectively. Let I be the incenter of $\triangle ABC$. We will now present a few properties of this configuration.



Property 25.1.1. The incenter I of $\triangle ABC$ is the midpoint of PQ.

Proof. We already proved this property in Property 17.7. ■



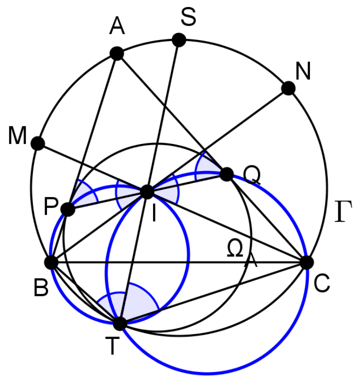


Property 25.1.2. The ray TI passes through the midpoint of $\widehat{BAC}$, i.e. TI is the angle bisector of $\angle BTC$.

Proof. Let's take a look at $\triangle TPQ$. By Property 25.1.1, TI is the $T-$median in that triangle. Since PA and QA are tangents to Ω_A, by Property 13.3, TA is the $T-$symmedian in that triangle. Therefore, $\angle PTA = \angle QTI$. Let the intersections of TP, TQ and TI with Γ be M, N and S, respectively. From the proof of Property 25.1.1, we know that M and N are the midpoints of the arcs $\widehat{AB}$ and $\widehat{AC}$ and therefore, $\angle MTA = \frac{\gamma}{2}$ and $\angle CTN = \frac{\beta}{2}$. Now, $\angle NTS \equiv \angle QTI = \angle PTA \equiv \angle MTA = \frac{\gamma}{2}$.
$\angle CTI \equiv \angle CTS = \angle CTN + \angle NTS = \frac{\beta}{2} + \frac{\gamma}{2} = 90° - \frac{\alpha}{2} = \frac{180° - \alpha}{2} = \frac{\angle BTC}{2}$

$$\therefore \angle BTI = \angle CTI$$ ■

Property 25.1.3. The quadrilaterals $BTIP$ and $CTIQ$ are cyclic. Moreover, CI and BI are tangents to $(BTIP)$ and $(CTIQ)$, respectively.



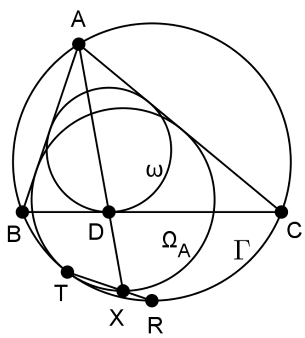
Proof. Since $\overline{PA} = \overline{QA}$ as tangent segments, from the isosceles $\triangle APQ$ we get $\angle API \equiv \angle APQ = 90^\circ - \frac{\alpha}{2}$. From Property 25.1.2, we have $\angle BTI = 90^\circ - \frac{\alpha}{2}$. Therefore, $\angle BTI = \angle API$, so $BTIP$ is cyclic. Similarly, $CTIQ$ is cyclic. □

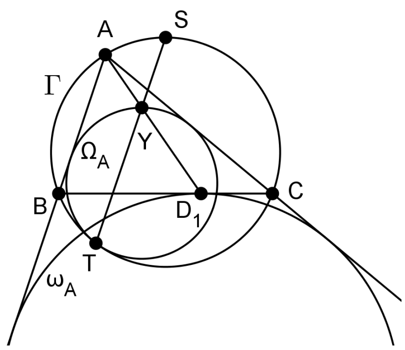
Let $CI \cap \Gamma = M$. From $\triangle BIC$ we know that $\angle BIC = 90^\circ + \frac{\alpha}{2}$. Therefore, $\angle BIM = 90^\circ - \frac{\alpha}{2} = \angle BTI$, so CI is tangent to $(BTIP)$. Similarly, BI is tangent to $(CTIQ)$. ■

Property 25.1.4. Let D and D_1 be the tangent points of BC with the incircle and $A-$excircle of $\triangle ABC$, respectively. Let R and S be the midpoints of the arcs $\widehat{BTC}$ and $\widehat{BAC}$ of Γ, respectively. Then, $TR \cap AD$, $TS \cap AD_1 \in \Omega_A$.

Proof. Let $X = TR \cap \Omega_A$ and let $\mathcal{X}_1$ be the homothety centered at T such that $\mathcal{X}_1 : \Gamma \to \Omega_A$. Then, by the definition of X, we have $\mathcal{X}_1 : R \to X$. Therefore, since R is the "bottom-most" point of Γ, X will be the "bottom-most" point of Ω_A, i.e. the tangent to Ω_A at X is parallel to BC.

Let ω be the incircle of $\triangle ABC$ and let $\mathcal{X}_2$ be the homothety centered at A such that $\mathcal{X}_2 : \Omega_A \to \omega$. Then, since D is the "bottom-most" point of ω, we have that $\mathcal{X}_2 : X \to D$ and therefore $A - D - X$ are collinear. □

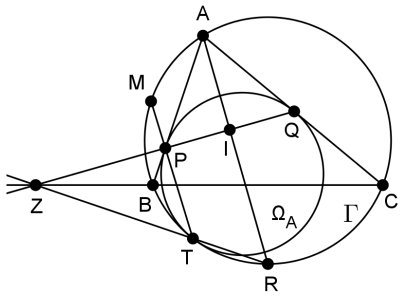




Similarly, if $Y = TS \cap \Omega_A$, we have $\mathcal{X}_1 : S \to Y$. Therefore, since S is the "top-most" point of Γ, Y will be the "top-most" point of Ω_A, i.e. the tangent to Ω_A at Y is parallel to BC.

Let ω_A be the $A-$excircle of $\triangle ABC$ and let $\mathcal{X}_3$ be the homothety centered at A such that $\mathcal{X}_3 : \Omega_A \to \omega_A$. Then, since D_1 is the "top-most" point of ω_A, we have that $\mathcal{X}_3 : Y \to D_1$ and therefore $A - Y - D_1$ are collinear. ■

Property 25.1.5. The lines PQ, BC and TR are concurrent, where R is the midpoint of the arc $\widehat{BC}$ of Γ that doesn't contain A.

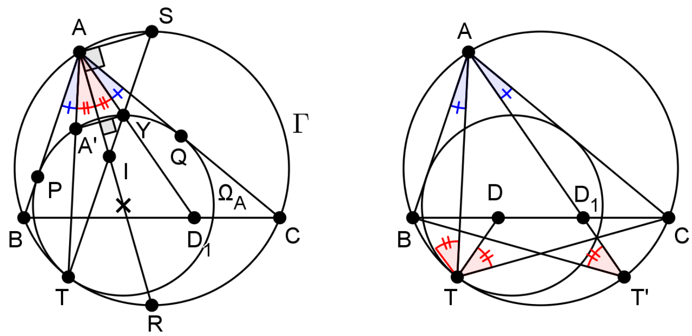


Proof. Let M be the midpoint of $\widehat{AB}$ and let $Z = TR \cap BC$. By Pascal's Theorem for the hexagon $TRABCM$, we get that the points $TR \cap BC = Z$, $RA \cap CM = I$ and $AB \cap MT = P$ are collinear, i.e. $Z \in IP \equiv PQ$. ■

Property 25.1.6. Let D and D_1 be the tangent points of BC with the incircle and $A-$excircle of $\triangle ABC$, respectively. Then,

$$\angle BAT = \angle CAD_1 \quad \text{and} \quad \angle BTA = \angle CTD$$

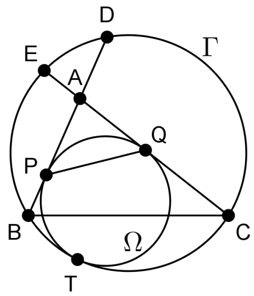
Proof. Let S and R be the midpoints of $\widehat{BAC}$ and $\widehat{BTC}$, $Y = TS \cap \Omega_A$ and $A' = TA \cap \Omega_A$. Let $\mathcal{X}_1$ be the homothety centered at T such that $\mathcal{X}_1 : \Gamma \to \Omega_A$. Then, $\mathcal{X}_1 : S \to Y$ and $\mathcal{X}_1 : A \to A'$ and $AS \parallel A'Y$. Since SR is diameter in Γ, we have $\angle SAR = 90°$ and therefore $A'Y \perp AR$. Also, the center of Ω_A lies on the angle bisector of $\angle PAQ \equiv \angle BAC$ which is AR. Therefore, AR is the side bisector of $A'Y$ and therefore $\overline{AA'} = \overline{AY}$. Now, AR is altitude in the isosceles $\triangle AA'Y$, so it is also the angle bisector of $\angle A'AY$. Therefore, $\angle BAA' = \angle CAY$. From Property 25.1.4, we know that $A-Y-D_1$ are collinear, so $\angle BAT \equiv \angle BAA' = \angle CAY \equiv \angle CAD_1$. □



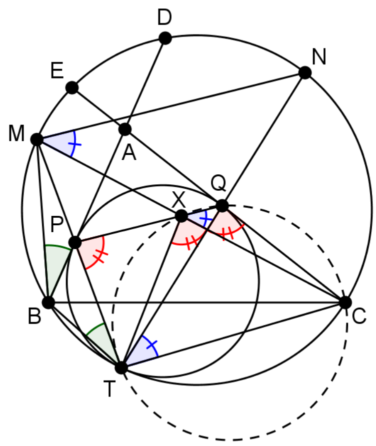
Let T' be the reflection of T with respect to the side bisector of BC. Then, $\widehat{CT'} = \widehat{BT}$ and therefore $\angle CAT' = \angle BAT = \angle CAD_1$, i.e. $A - D_1 - T'$ are collinear. From Property 10.1.3, we know that the reflection of D w.r.t. the midpoint of BC is D_1. Therefore, $\angle CTD = \angle BT'D_1 \equiv \angle BT'A = \angle BTA$ ■

25.2 Curvilinear Incircles

Let ABC be a triangle with incenter I and let Γ be a circle trough the points B and C, so that A is inside Γ. Let D and E be the second intersections of BA and CA with Γ. Let Ω be a circle tangent to the segments AB, AC and Γ at the points P, Q and T, respectively. Then, Ω is said to be a *curvilinear incircle* for $\triangle BCD$ (or $\triangle BCE$).

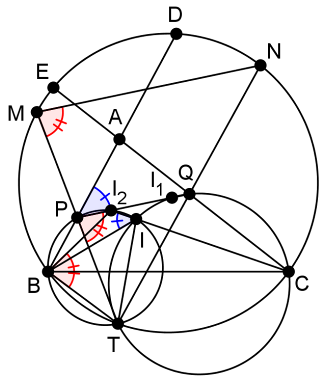


Property 25.2.1 (Sawayama Lemma)**.** Let I_1 and I_2 be the incenters of $\triangle BCD$ and $\triangle BCE$. Then $I_1, I_2 \in PQ$.



Proof. Let TP and TQ intersect Γ again at M and N, respectively, and let $X = CM \cap PQ$. Then, from the homothety that sends (TPQ) to (TMN), because they are tangent to each other, we get that M is the midpoint of $\widehat{BD}$ and $MN \parallel PQ$. Therefore, $\angle CXQ = \angle CMN = \angle CTN \equiv \angle CTQ$, so $CTXQ$ is cyclic. From this cyclic quadrilateral and because CQ is tangent to (TPQ), we get $\angle TXC = \angle TQC = \angle TPQ$. Therefore, $CX \equiv MX$ is tangent to (TPX), so $\overline{MX}^2 = \overline{MP} \cdot \overline{MT}$. Since M is the midpoint of $\widehat{BD}$, $\angle MBP \equiv \angle MBD = \angle MTB$, so $\triangle MBP \sim \triangle MTB$, and therefore $\overline{MB}^2 = \overline{MP} \cdot \overline{MT}$. Thus, $\overline{MX} = \overline{MB}$. From Property 7.2, since CM is the angle bisector of $\angle BCD$, we get that $X \equiv I_1$, i.e. $I_1 \in PQ$. Similarly, $I_2 \in PQ$. ∎

Property 25.2.2. Let I_1 and I_2 be the incenters of $\triangle BCD$ and $\triangle BCE$. Then BPI_2IT and CQI_1IT are cyclic.



Proof. Since I_2 lies on the angle bisector of $\angle BCE \equiv \angle BCA$, the points $C - I - I_2$ are collinear. Then, because $\angle BIC = 90^\circ + \frac{\alpha}{2}$, we get $\angle BII_2 = 90^\circ - \frac{\alpha}{2}$. From Property 25.2.1, we know that $P - I_2 - Q$ are collinear. Since $\overline{AP} = \overline{AQ}$, we get $\angle API_2 \equiv \angle APQ = 90^\circ - \frac{\alpha}{2}$. Thus, $\angle BII_2 = \angle API_2$, so BII_2P is cyclic.

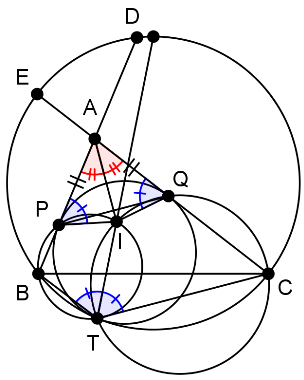
Let TP and TQ intersect Γ again at M and N, respectively. Then, from the homothety that sends (TPQ) to (TMN), because they are tangent to each other, we get that M and N are midpoints of $\widehat{BD}$ and $\widehat{CE}$, i.e. $I_1 \in CM$ and $I_2 \in BN$. From the homothety, we also get $MN \parallel PQ$. Now,

$$\angle I_2PT \equiv \angle QPT = \angle NMT = \angle NBT \equiv \angle I_2BT,$$

so BPI_2T is cyclic.

In conclusion, BPI_2IT is cyclic. Similarly, CQI_1IT is cyclic. ■

Property 25.2.3. The ray TI passes through the midpoint of $\widehat{BC}$ (that doesn't contain T), i.e. TI is the angle bisector of $\angle BTC$.

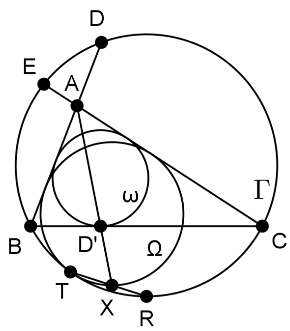


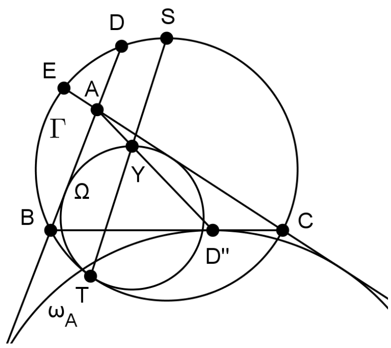
Proof. From Property 25.2.2, we know that $BPIT$ and $CQIT$ are cyclic. Therefore $\angle BTI = \angle IPA$ and $\angle CTI = \angle IQA$. Since I lies on the angle bisector of $\angle PAQ$ and $\overline{AP} = \overline{AQ}$ as tangent segments, then by SAS $\triangle IAP \cong \triangle IAQ$ and therefore $\angle IPA = \angle IQA$. By combining these three angle equalities, we get $\angle BTI = \angle CTI$. ■

Property 25.2.4. Let D' and D'' be the tangent points of BC with the incircle and $A-$excircle of $\triangle ABC$, respectively. Let R and S be the midpoints of the arcs $\widehat{BTC}$ and $\widehat{BEDC}$ of Γ, respectively. Then, $TR \cap AD'$, $TS \cap AD'' \in \Omega$.

Proof. Let $X = TR \cap \Omega$ and let $\mathcal{X}_1$ be the homothety centered at T such that $\mathcal{X}_1 : \Gamma \to \Omega$. Then, by the definition of X, we have $\mathcal{X}_1 : R \to X$. Therefore, since R is the "bottom-most" point of Γ, X will be the "bottom-most" point of Ω, i.e. the tangent to Ω at X is parallel to BC.

Let ω be the incircle of $\triangle ABC$ and let $\mathcal{X}_2$ be the homothety centered at A such that $\mathcal{X}_2 : \Omega \to \omega$. Then, since D' is the "bottom-most" point of ω, we have that $\mathcal{X}_2 : X \to D'$ and therefore $A - D' - X$ are collinear. $\square$

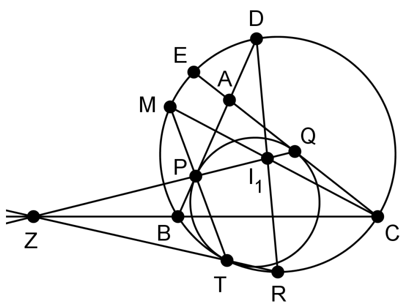




Similarly, if $Y = TS \cap \Omega$, we have $\mathcal{X}_1 : S \to Y$. Therefore, since S is the "top-most" point of Γ, Y will be the "top-most" point of Ω, i.e. the tangent to Ω at Y is parallel to BC.

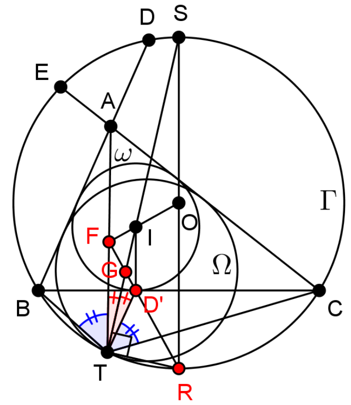
Let ω_A be the $A-$excircle of $\triangle ABC$ and let $\mathcal{X}_3$ be the homothety centered at A such that $\mathcal{X}_3 : \Omega \to \omega_A$. Then, since D'' is the "top-most" point of ω_A, we have that $\mathcal{X}_3 : Y \to D''$ and therefore $A - Y - D''$ are collinear. $\blacksquare$

Property 25.2.5. The lines PQ, BC and TR are concurrent, where R is the midpoint of the arc $\widehat{BTC}$ of Γ.



Proof. Let M be the midpoint of $\widehat{BED}$ and let $Z = TR \cap BC$. By Pascal's Theorem for the hexagon $TRDBCM$, we get that the points $TR \cap BC = Z$, $RD \cap CM = I_1$ and $DB \cap MT = P$ are collinear, i.e. $Z \in I_1P \equiv PQ$. $\blacksquare$

Property 25.2.6. Let D' be the tangent point of BC with the incircle of $\triangle ABC$, ω. Then, $\angle BTA = \angle CTD'$.

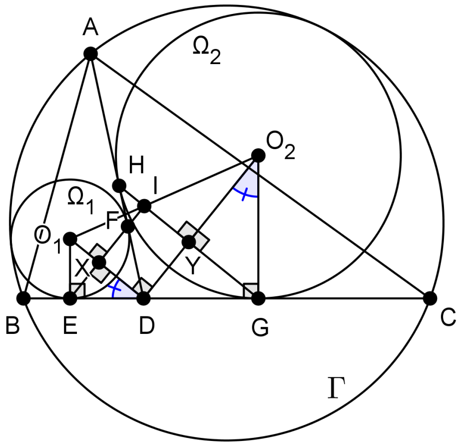


Proof. Let R and S be the midpoints of arcs $\widehat{BTC}$ and $\widehat{BEDC}$, respectively. From Property 25.2.3 we know that $S \in TI$, i.e. TS is angle bisector of $\angle BTC$. Therefore, we need to prove that TS is also the angle bisector of $\angle ATD'$. Since $TR \perp TS$, by Property 22.3, we need to prove that the pencil $T(S, R; A, D')$ is harmonic.

Let's take a look at the three circles Γ, Ω, and ω. We know that A is the exsimilicenter of ω and Ω; and T is the exsimilicenter of Ω and Γ. Therefore, by Property 17.8, the exsimilicenter F of ω and Γ lies on AT. On the other hand, F lies on the line connecting the centers of ω and Γ, i.e. $F \in IO$. Since D' is sent to R by $\mathcal{X}_F : \omega \to \Gamma$ (as the "bottom-most" point), we get $F \in D'R$.

Finally, since $\overline{SO} = \overline{OR}$ and $ID' \parallel SR$, by Property 22.7, we get that the pencil $I(S, R; O, D')$ is harmonic. By Property 22.6, considering the line RD', we get that $(G, R; F, D')$ is harmonic, where $G = IS \cap RD'$. Therefore, $T(G, R; F, D') \equiv T(S, R; A, D')$ is a harmonic pencil. ■

Property 25.2.7 (Sawayama Thebault's Theorem)**.** Let ABC be a triangle with circumcircle Γ and incenter I. Let $D \in BC$. Let Ω_1 be the circle tangent to the line segments DA and DB and to the circle Γ, and let Ω_2 be the circle tangent to the line segments DA and DC and to the circle Γ. If O_1 and O_2 are the centers of Ω_1 and Ω_2, respectively, prove that $O_1 - I - O_2$ are collinear.



Proof. Let the tangent points of Ω_1 with BC and AD be E and F, respectively. Let the tangent points of Ω_2 with BC and AD be G and H, respectively. By Property 25.2.1, we get that $I \in EF$ and $I \in GH$.

Let's take a look at the quadrilateral O_1EDF. We have $\overline{DE} = \overline{DF}$ as tangent segments and $\overline{O_1E} = \overline{O_1F}$ as radii. Therefore, O_1EDF is a kite and thus $O_1D \perp EF$ and DO_1 is angle bisector of $\angle EDF \equiv \angle BDA$. Similarly, $O_2D \perp GH$ and DO_2 is angle bisector of $\angle GDH \equiv \angle CDA$. Therefore, $\angle O_1DO_2 = \frac{\angle BDC}{2} = 90°$. By combining this result with the aforementioned perpendicularities, we get $DO_1 \parallel GH$ and $DO_2 \parallel EF$. Let $EF \cap DO_1 = X$ and $GH \cap DO_2 = Y$. Now, since we want to prove that $I \in O_1O_2$, let $EF \cap O_1O_2 = I'$ and let $GH \cap O_1O_2 = I''$. Then, by Thales' Proportionality Theorem, we get

$$\frac{\overline{O_1I'}}{\overline{I'O_2}} = \frac{\overline{O_1X}}{\overline{XD}} \quad \text{and} \quad \frac{\overline{O_1I''}}{\overline{I''O_2}} = \frac{\overline{DY}}{\overline{YO_2}}.$$

Since $\angle O_1ED = 90° = \angle O_2GD$ and $\angle EDO_1 = 90° - \angle GDO_2 = \angle GO_2D$, by AA we get $\triangle O_1ED \sim \triangle DGO_2$ and the points X and Y are the corresponding feet of perpendiculars from the right angle to the hypotenuse in these triangles, so

$$\frac{\overline{O_1X}}{\overline{XD}} = \frac{\overline{DY}}{\overline{YO_2}}$$

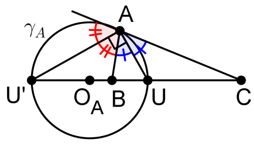
By combining the last three proportions, we get that $I' \equiv I''$, which means that $EF \cap GH \in O_1O_2$, i.e. $I \in O_1O_2$. ■

Related problems: 166, 209, 215, 221 and 223.

Chapter 26

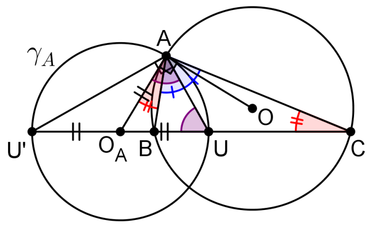
Apollonian Circles

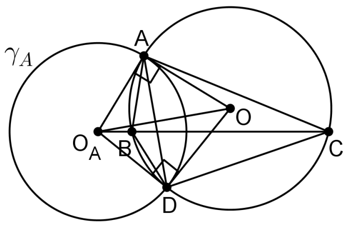
In Property 22.2, we already saw that the locus of a point the ratio of whose distances from two fixed points is constant is a circle, called *circle of Apollonius*. We know define the *Apollonian circles* (or *circles of Apollonius*) of a triangle. If U and U' are the feet of the $A-$internal and $A-$external angle bisector on the line BC, then the circle with diameter UU' is called the $A-$Apollonian circle of $\triangle ABC$, γ_A. Since $\angle UAU'$ is the sum of two supplementary angles halved, we get that $A \in \gamma_A$. So, for any point $P \in \gamma_A$, we have $\frac{\overline{PB}}{\overline{PC}} = \frac{\overline{AB}}{\overline{AC}}$.



Property 26.1. The Apollonian circles are orthogonal to the circumcircle.

Proof. Let O_A and O be the circumcenters of γ_A and (ABC), respectively. We need to prove that $\angle O_AAO = 90°$, i.e. that O_AA is tangent to (ABC). Since O_A is midpoint of the hypotenuse UU' in the right $\triangle UAU'$, we have $\overline{O_AA} = \overline{O_AU}$. Therefore $\angle O_AAU = \angle O_AUA$, i.e. $\angle O_AB + \angle BAU = \angle UAC + \angle ACU$, and since AU is angle bisector of $\angle BAC$, we get $\angle O_AAB = \angle ACB$, so O_AA is tangent to (ABC). ■



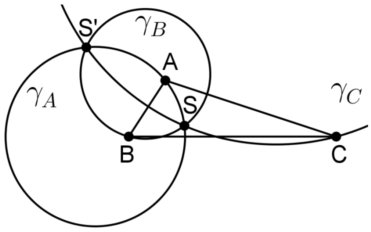


Property 26.2. The common chord of the circumcircle and each of the Apollonian circles is a symmedian in $\triangle ABC$.

Proof. Let D be the second intersection of (ABC) and γ_A. In Property 26.1 we already showed that O_AA is tangent to (ABC). Since O_AO is side bisector of AD, we get that O_AD is also tangent to (ABC). Since the tangents to (ABC) at A and D concur with BC, by Property 13.3, we get that CB is a symmedian in $\triangle CAD$. Therefore, by Property 13.5, we get that $ABDC$ is a harmonic quadrilateral and that AD is symmedian in $\triangle ABC$. ■

Property 26.3. The three Apollonian circles are coaxial.

Proof. Let γ_A and γ_B intersect at S (inside $\triangle ABC$) and S' (outside $\triangle ABC$).



$$\frac{\overline{SB}}{\overline{SC}} = \frac{\overline{AB}}{\overline{AC}} \ (\because S \in \gamma_A)$$

$$\frac{\overline{SC}}{\overline{SA}} = \frac{\overline{BC}}{\overline{BA}} \ (\because S \in \gamma_B).$$

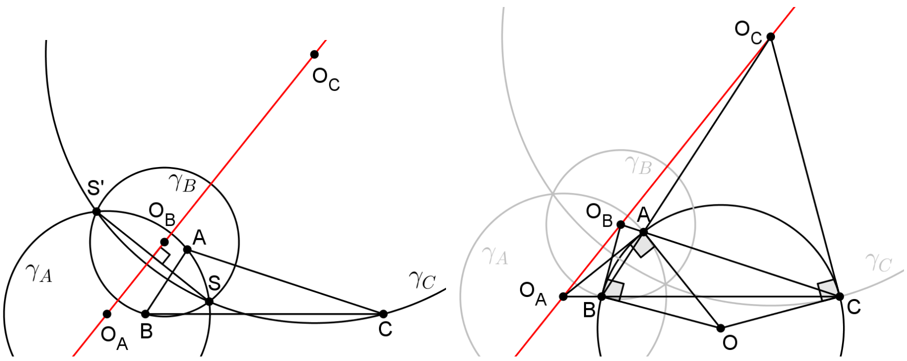
By multiplying these equations, we get $\frac{\overline{SB}}{\overline{SA}} = \frac{\overline{CB}}{\overline{CA}}$, i.e. $S \in \gamma_C$. Similarly, $S' \in \gamma_C$.

Therefore, SS' is the radical axis of the three Apollonian circles. ■

Remark. The points S and S' are known as the first and second isodynamic points of $\triangle ABC$, respectively.

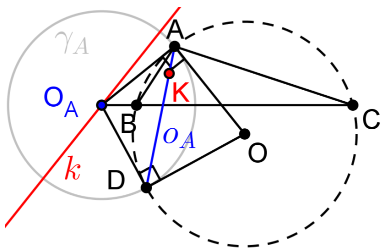
Property 26.4. The centers of the three Apollonian circles are collinear.

Proof 1. From Property 26.3 we know that SS' is a common chord of three Apollonian circles. Therefore, their centers all lie on the side bisector of SS'. ■



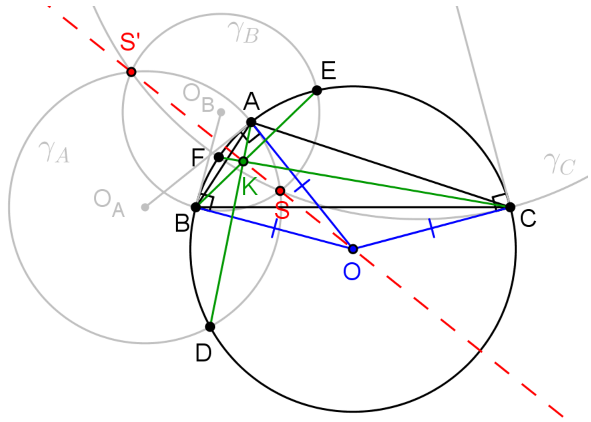
Proof 2. From Property 26.1 we know that $O_A = AA \cap BC$. By Pascal's Theorem for $AABBCC$, we get that O_A, O_B, O_C are collinear. ■

Proof 3. From Property 26.3 we know that O_AA and O_AD are tangents to (ABC), so by Property 21.1 we get that AD is the polar of O_A w.r.t. (ABC). From Property 26.2 we know that AD is a symmedian in $\triangle ABC$, so the Lemoine Point K lies on AD. By La Hire's Theorem, since $K \in o_A$, we get that $O_A \in k$. Similarly, O_B and O_C also lie on the polar of K w.r.t. (ABC) and therefore the three centers are collinear. ■



Remark. This is why the line that contains the three centers of the Apollonian circles is known as the *Lemoine line*, or *Lemoine axis*.

Property 26.5. The circumcenter O and the Lemoine Point K of $\triangle ABC$ lie on the radical axis of the three Apollonian circles, SS'.



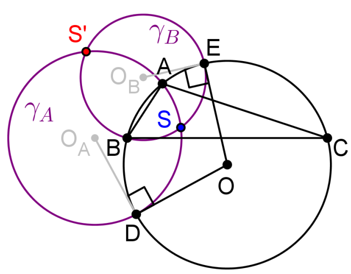
Proof. From Property 26.1 we know that OA is tangent to γ_A, so the power of the point O w.r.t. γ_A is $\overline{OA}^2$. Since $\overline{OA} = \overline{OB} = \overline{OC}$ as circumradii in (ABC), we get that O has equal power with respect to $\gamma_A, \gamma_B, \gamma_C$, so it lies on their radical axis SS'.

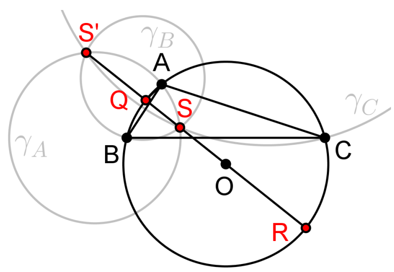
From Property 26.2 we know that the symmedian AD is a common chord of (ABC) and γ_A, and since $K \in AD$, it has equal power with respect to these two circles. Since K lies on all three symmedians, we get that it has equal power with respect to (ABC), $\gamma_A, \gamma_B, \gamma_C$, so it lies on SS'. ∎

Remark. This line is known as the *Brocard axis* of $\triangle ABC$.

Property 26.6. The isodynamic points S and S' are inverses of each other with respect to (ABC).

Proof. Recall that under inversion, a circle orthogonal to the circle of inversion is sent to itself. By Property 26.1, we get that $\mathcal{J}_O : \gamma_A \leftrightarrow \gamma_A$. Therefore, the point $S \in \gamma_A$ is sent to some point on $\gamma_A' \equiv \gamma_A$, but not to itself since $S \notin (ABC)$. Similarly, since $S \in \gamma_B$ and $\mathcal{J}_O : \gamma_B \leftrightarrow \gamma_B$, S must be sent to a point on γ_B. The only such point is S', i.e. $\mathcal{J}_O : S \leftrightarrow S'$. ∎

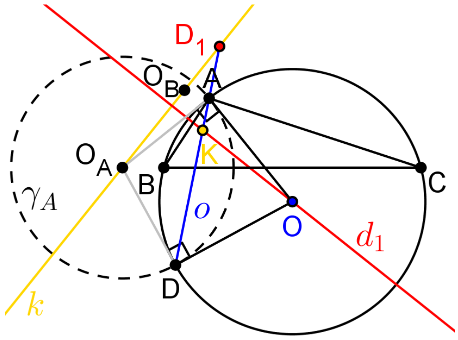




Property 26.7. Let Q, R be the intersections of SS' with (ABC). Then, $(Q, R; S, S')$ is a harmonic division.

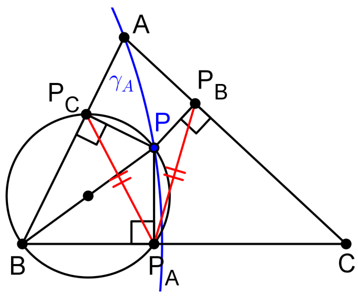
Proof. From Property 26.6 we know that $\mathcal{J}_O : S \leftrightarrow S'$. From Property 26.5 we get that QR is diameter in (ABC), so by Property 22.4 we get that $(Q, R; S, S')$ is a harmonic division. ∎

Property 26.8. Each symmedian intersect the Lemoine axis at the pole of the Brocard axis with respect to the corresponding Apollonian circle.



Proof. Let the symmedian AD intersect the Lemoine axis O_AO_B at D_1. From Property 26.1 we get that OA and OD are tangent to γ_A, so AD is the polar of O with respect to γ_A, i.e. $AD \equiv o$. Since $D_1 \in o$, by La Hire's Theorem, we get that $O \in d_1$. On the other hand, from Proof 3 of Property 26.4, we know that O_AO_B is the polar of the Lemoine point K. Since $D_1 \in k$, we have $K \in d_1$. In conclusion, we got that $OK \equiv d_1$, i.e. D_1 is the pole of the Brocard axis OK with respect to γ_A. ■

Property 26.9. The locus of the points P such that the pedal triangle $\triangle P_AP_BP_C$ is P_A-isosceles is the $A-$Apollonian circle.



Proof. Let P be a point with pedal triangle $\triangle P_AP_BP_C$. Using the Law of Sines in $\triangle P_AP_CB$, we get $\overline{P_AP_C} = 2R_{\triangle P_AP_CB} \cdot \sin\beta$. But, $\angle PP_AB = 90° = \angle PP_CB$ implies that PP_ABP_C is cyclic with diameter PB, so $\overline{P_AP_C} = \overline{PB} \cdot \sin\beta$. Similarly, $\overline{P_AP_B} = \overline{PC} \cdot \sin\gamma$. By dividing these equations, and by using Law of Sines in $\triangle ABC$, we get:

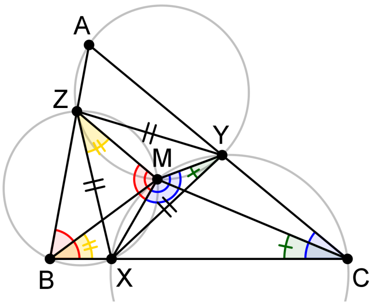
$$\overline{P_AP_B} = \overline{P_AP_C} \iff \frac{\overline{PB}}{\overline{PC}} = \frac{\sin\gamma}{\sin\beta} = \frac{\overline{AB}}{\overline{AC}} \iff P \in \gamma_A \qquad ■$$

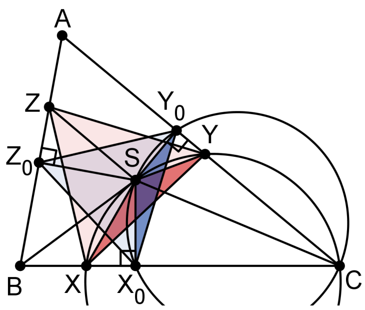
Property 26.10. The pedal triangle of the first isodynamic point S has the least area of all equilateral triangles having vertices on all three sides of a triangle.

Proof. Let $\triangle XYZ$ be an arbitrary equilateral triangle such that $X \in BC$, $Y \in CA$ and $Z \in AB$. By Miquel's Theorem for $\triangle ABC$ and points X, Y, Z on its sidelines, we get that $(AYZ), (BXZ), (CXY)$ are concurrent at a point M. We will prove that $M \equiv S$. Using $\overline{XY} = \overline{XZ}$ and Law of Sines in $\triangle MBC$, $\triangle MXY$, $\triangle MXZ$ and $\triangle ABC$, we get:

$$\begin{aligned}\frac{\overline{MB}}{\overline{MC}} &= \frac{\sin(\angle MCB)}{\sin(\angle MBC)} = \frac{\sin(\angle MYX)/\overline{MX}}{\sin(\angle MZX)/\overline{MX}} = \frac{\sin(\angle YMX)/\overline{XY}}{\sin(\angle ZMX)/\overline{XZ}} = \\ &= \frac{\sin(180^\circ - \gamma)}{\sin(180^\circ - \beta)} = \frac{\sin\gamma}{\sin\beta} = \frac{\overline{AB}}{\overline{AC}} \iff M \in \gamma_A\end{aligned}$$

Similarly, $M \in \gamma_B, \gamma_C$ and thus $M \equiv S$.





Let $\triangle X_0Y_0Z_0$ be the pedal triangle of S. Since $S \in \gamma_A, \gamma_B$, by Property 26.9, we get that $\triangle X_0Y_0Z_0$ is equilateral. Notice that $X_0X \cap Y_0Y = C$ and $S \in (CX_0Y_0), (CXY)$, so by Property 18.4, S is the center of the spiral similarity that sends X_0Y_0 to XY. Therefore, $\triangle SX_0Y_0 \sim \triangle SXY$. Since $\overline{SX} \geq \overline{SX_0}$, we get:

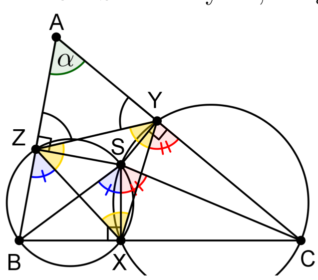
$$\frac{\overline{XY}}{\overline{X_0Y_0}} = \frac{\overline{SX}}{\overline{SX_0}} \geq 1 \iff \overline{XY} \geq \overline{X_0Y_0}.$$

Of all the equilateral triangles $\triangle XYZ$, the one with the least area is the one with the shortest side XY, which is $\triangle X_0Y_0Z_0$ with its side X_0Y_0. ∎

Property 26.11. The First isodynamic point S is the isogonal conjugate of the First Fermat Point.

Proof. Let $\triangle XYZ$ be the pedal triangle of S. From Property 26.9, we know that $\triangle XYZ$ is equilateral. Also, since $BXSZ$ and $CXSY$ are cyclic, we get:

$$\begin{aligned}\angle BSC &= \angle BSX + \angle XSC = \\ &= \angle BZX + \angle XYC = \\ &= 180^\circ - \angle XZY - \angle YZA + \\ &\quad + 180^\circ - \angle XYZ - \angle ZYA = \\ &= (180^\circ - \angle YZA - \angle ZYA) + 60^\circ = \\ &= \angle BAC + 60^\circ\end{aligned}$$

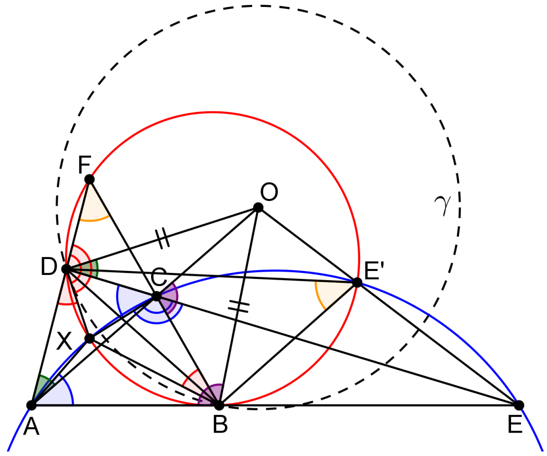


Let S^* be the isogonal conjugate of S. From Property 24.1.6, we know that $\angle BSC + \angle BS^*C = \angle BAC + 180°$. Therefore, $\angle BS^*C = 120°$. Similarly, we can get $\angle CS^*A = \angle AS^*C = 120°$. But, from the proof of Property 10.9.1, we know that the First Fermat Point is the only such point. ∎

Remark. Similarly, it can be proven that the Second isodynamic point S' is the isogonal conjugate of the Second Fermat Point.

Example 26.1 (IMO 2018/6)**.** A convex quadrilateral $ABCD$ satisfies $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$. Point X lies inside $ABCD$ so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle AXB + \angle CXD = 180°$.

Proof. WLOG assume that the rays AB and DC intersect at E, while the rays BC and AD intersect at F. We have $\angle XAB = \angle XCD = 180° - \angle XCE$ and $\angle XBC = \angle XDA = 180° - \angle XDF$, so $AXCE$ and $BXDF$ are cyclic.



By definition, γ, the $B-$Apollonian circle of $\triangle ABC$, has center O that lies on AC. Also, it is the locus of the points P such that $\frac{\overline{PA}}{\overline{PC}} = \frac{\overline{BA}}{\overline{BC}}$. Since $\frac{\overline{BA}}{\overline{BC}} = \frac{\overline{DA}}{\overline{DC}}$, we get $D \in \gamma$, i.e. γ is also the $D-$Apollonian circle of $\triangle ACD$. By Property 26.1, we get that OB is tangent to (ABC) and OD is tangent to (ACD). Let $\mathcal{J}_{O,\overline{OB}} : E \leftrightarrow E'$. Then, $\overline{OE} \cdot \overline{OE'} = r^2 = \overline{OB}^2 = \overline{OA} \cdot \overline{OC}$. Therefore, $E' \in (AXCE)$.

On the other hand, using Property 20.1 and Property 5.3, we get:

$$\angle BE'D = \angle OE'B - \angle OE'D = \angle OBE - \angle ODE \equiv 180° - \angle OBA - \angle ODC =$$
$$= 180° - \angle OCB - \angle OAD \equiv \angle ACB - \angle CAF = \angle CFA \equiv \angle BFD,$$

so $E' \in (BXDF)$.

Finally, we will show that $\angle AXB + \angle CXD = 180°$ by angle chasing each of them. At the end we will use that $\triangle OBD$ is isosceles, i.e. $\angle BDO = \angle DBO = \varphi$, since $\overline{OB}^2 = \overline{OA} \cdot \overline{OC} = \overline{OD}^2$.

$$\angle AXB = \angle AXE' - \angle BXE' = 180° - \angle E'EA - \angle BDE' =$$
$$= 180° - \angle OEB - (\angle BDO - \angle ODE') = 180° - \angle OEB - \varphi + \angle OED$$
$$\angle CXD = \angle E'XD - \angle E'XC = \angle E'BD - \angle E'EC =$$
$$= \angle DBO + \angle OBE' - \angle OED = \varphi + \angle OEB - \angle OED$$

∎

Chapter 27

Apollonius' Problem

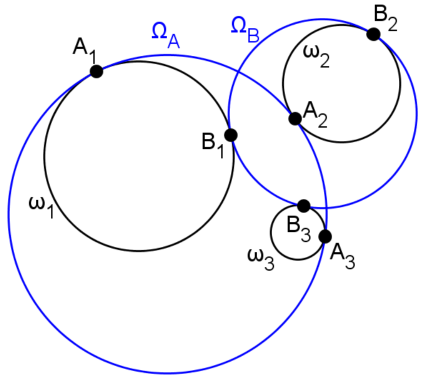
This topic is not directly related to Olympiad geometry problems, but it is a nice collection of properties that we already visited during our journey combined in a beautiful result.

Example 27.1 (Apollonius' problem)**.** Construct circles that are tangent to three given circles in a plane, ω_1, ω_2 and ω_3.

We will examine the case where the three circles are in general position, i.e. none of them intersect and all of them have different radii.

Firstly, let's find the number of solution circles that are tangent to all three circles. The solution circles can be tangent either internally or externally to any of the three circles. So the number of solution circles is $2^3 = 8$.

We will explain Gergonne's approach to solving this problem. It considers the solution circles in pairs such that if one of the solution circles is internally tangent to a given circle, then the other solution circle is externally tangent to that circle and vice versa. For example, if a solution circle is internally tangent to ω_1 and ω_3, but externally tangent to ω_2, then the paired solution circle is externally tangent to ω_1 and ω_3, but internally tangent to ω_2.



Let Ω_A and Ω_B be a pair of solution circles. Let Ω_A be tangent to ω_1, ω_2 and ω_3 at A_1, A_2 and A_3. Let Ω_B be tangent to ω_1, ω_2 and ω_3 at B_1, B_2 and B_3.

So, we somehow need to find these 6 tangent points. Then, the circumcircles of $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$ would be the solution circles. Gergonne's approach was to construct a line l_1 such that A_1 and B_1 must always lie on it. Then, A_1

and B_1 could be obtained as the intersection points of l_1 and ω_1. Similarly, by finding lines l_2 and l_3 that contained A_2 and B_2, and A_3 and B_3, respectively, we would find all 6 tangent points.

Let's recall, from section 14.2, that the radical center of three circles is the center of the unique circle (called the radical circle) that intersects the three given circles orthogonally. Let R be the radical center of ω_1, ω_2 and ω_3. Now, consider the inversion $\mathcal{J}$ with the radical circle as the circle of inversion. Since the radical circle is orthogonal to the three given circles, each of them will be sent to itself. Since the solution circles are tangent to the three given circles, their images need to be tangent to the images of the given circles (which happen to be the three circles themselves), so the solution circles will be sent one into the other, i.e.

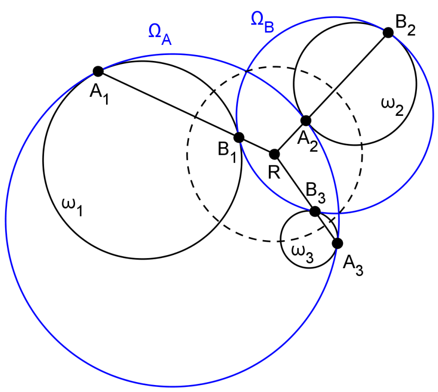
$$\mathcal{J} : \omega_1 \leftrightarrow \omega_1, \quad \mathcal{J} : \omega_2 \leftrightarrow \omega_2, \quad \mathcal{J} : \omega_3 \leftrightarrow \omega_3,$$

$$\mathcal{J} : \Omega_A \leftrightarrow \Omega_B.$$

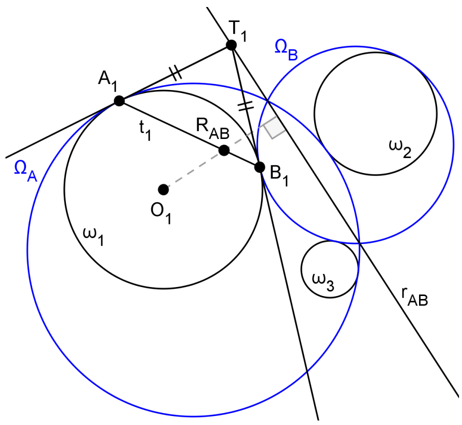
Therefore, the tangent point $A_1 = \omega_1 \cap \Omega_A$ will be sent to a point

$$A_1' = \omega_1' \cap \Omega_A' = \omega_1 \cap \Omega_B = B_1, \text{ i.e. } \mathcal{J} : A_1 \leftrightarrow B_1.$$

Since the center of inversion, the original point and the image point are collinear, we get that the radical center lies on the line A_1B_1, i.e. $R \in l_1$. Thus, we found one point on the line l_1. Now we need to find another one in order to be able to construct it.



(a) $R \in l_1$

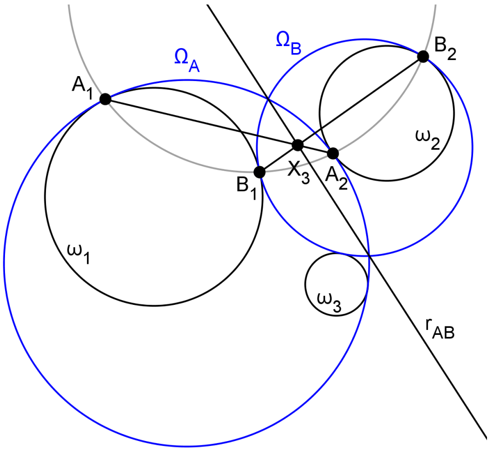


(b) $R_{AB} \in l_1$

Let the tangents to ω_1 at A_1 and B_1 intersect at T_1. Then, $\overline{T_1A_1} = \overline{T_1B_1}$ and also, by Property 21.1, A_1B_1 is the polar of T_1 with respect to ω_1, i.e. $A_1B_1 \equiv t_1$. Notice that T_1A_1 and T_1B_1 are also tangents to the circles Ω_A and Ω_B. By Property 14.2 and since $\overline{T_1A_1} = \overline{T_1B_1}$, the power of the point T_1 with respect to Ω_A and Ω_B is equal, which means that T_1 lies on their radical axis r_{AB}. By La Hire's Theorem, the pole of r_{AB} with respect to ω_1 lies on t_1. So, here is our second point on the line $t_1 \equiv A_1B_1 \equiv l_1$.

But in order to construct the pole of r_{AB} with respect to ω_1, we firstly need to construct r_{AB}. How do we constuct the radical axis of two circles Ω_A and Ω_B if we don't have them yet? Well, we will find points that should lie on the radical axis and then we will construct the radical axis as the line through those points.

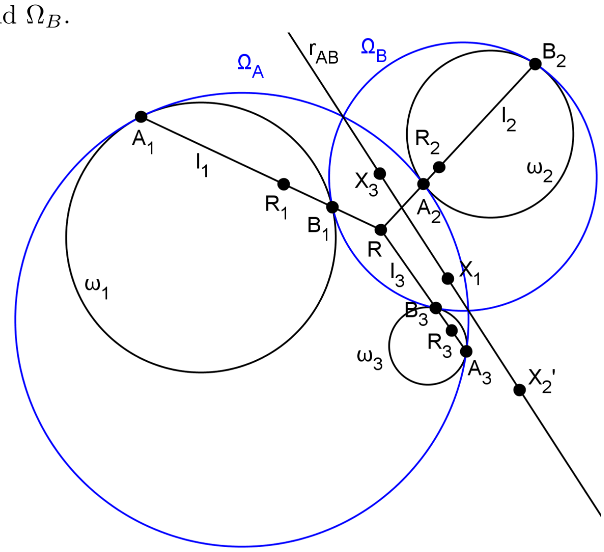
Let's recall, from Property 17.10 that if a circle is tangent to two other circles, then the line through the tangent points passes through one of the homothetic centers of the two circles. Let X_3 be a homothetic center of ω_1 and ω_2, which can be constructed as the intersection of their common tangents. Then $X_3 \in A_1A_2$ and $X_3 \in B_1B_2$.



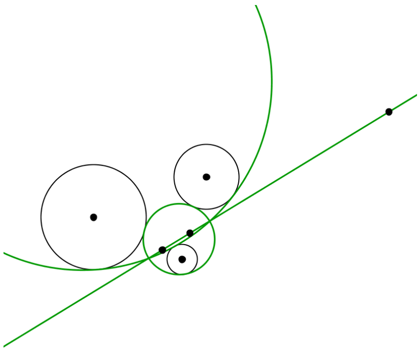
Recall also, from section 17.1, the definition and the properties of antihomologous points. This means that in our case, since Ω_A is tangent to ω_1 and ω_2, A_1 and A_2 are a pair of antihomologous points. Similarly, B_1 and B_2 are antihomologous points. Because, we know that two pairs of antihomologous points are concyclic, then:

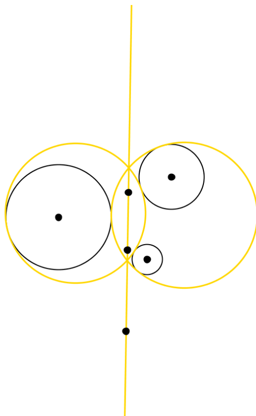
$$\overline{X_3A_1} \cdot \overline{X_3A_2} = \overline{X_3B_1} \cdot \overline{X_3B_2}$$

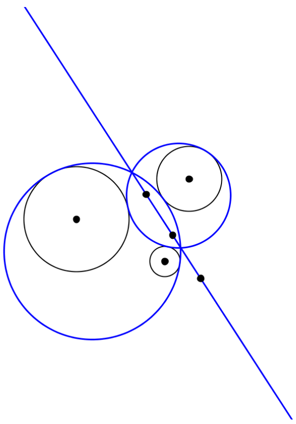
But since $A_1, A_2 \in \Omega_A$ and $B_1, B_2 \in \Omega_B$, this means that the power of the point X_3 with respect to Ω_A and Ω_B is the same, i.e. X_3 lies on their radical axis r_{AB}. Similarly, the homothetic centers X_1 and X_2 (which can be constructed as the intersection of the common tangents of ω_2 and ω_3, and ω_1 and ω_3, respectively) also lie r_{AB}. Thus, we know how to construct the radical axis r_{AB} of the solution circles Ω_A and Ω_B.

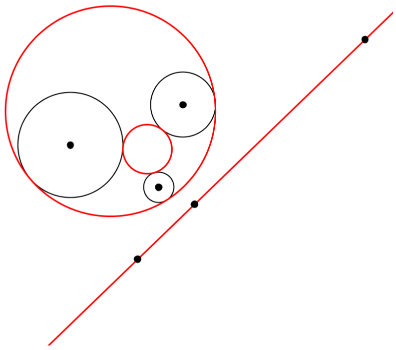


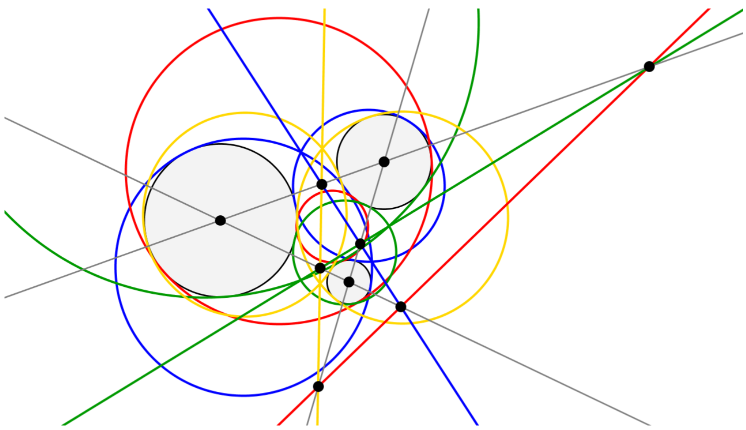
In summary, the desired line l_1 is defined by two points: the radical center R of the three given circles and the pole with respect to ω_1 of the line connecting the homothetic centers. Depending on whether we choose all three external homothetic centers (1 possibility), or we choose one external and the other two internal homothetic centers (3 possibilities), we have 4 ways of defining “the line connecting the homothetic centers”[1]. Each of these 4 lines generates a different pair of solution circles, so that’s how we can get all 8 solution circles.











[1]Recall Monge-d’Alembert Theorem, page 121

Part II

Mixed Problems

Problems

Problem 1. Let C be a point on the line segment AB. Let D be a point that doesn't lie on the line AB. Let M and N be points on the angle bisectors of $\angle ACD$ and $\angle BCD$, respectively, such that $MN \parallel AB$. Prove that the line CD bisects MN.

Problem 2. Let ABC be a triangle and let M be a point on the ray AB beyond B, such that $\overline{BM} = \overline{BC}$. Prove that MC is parallel to the angle bisector of $\angle ABC$.

Problem 3. Let ABC be an isosceles triangle ($\overline{CA} = \overline{CB}$). Let S_{XY} denote the side bisector of a segment XY. Let $S_{CA} \cap CB = P$ and $S_{CB} \cap CA = Q$. Prove that $PQ \parallel AB$.

Problem 4 (IGO 2015, Elementary)**.** Let ABC be a triangle with $\angle A = 60°$. The points M, N and K lie on BC, AC and AB, respectively, such that $\overline{BK} = \overline{KM} = \overline{MN} = \overline{NC}$. If $\overline{AN} = 2\overline{AK}$, find the values of $\angle B$ and $\angle C$.

Problem 5. Let BD be a median in $\triangle ABC$. The points E and F divide the median BD in three equal parts, such that $\overline{BE} = \overline{EF} = \overline{FD}$. If $\overline{AB} = 1$ and $\overline{AF} = \overline{AD}$, find the length of the line segment CE.

Problem 6. Let I be the incenter of $\triangle ABC$. Let l be a line through I, parallel to AB, that intersects the sides CA and CB at M and N, respectively. Prove that $\overline{AM} + \overline{BN} = \overline{MN}$.

Problem 7 (Serbia 2017, Opstinsko IIB)**.** The diagonals of a convex quadrilateral $ABCD$ intersect at O. Prove that the circumcenters of the triangles ABO, BCO, CDO and DAO are vertices of a parallelogram.

Problem 8 (IGO 2018, Elementary)**.** Convex hexagon $A_1A_2A_3A_4A_5A_6$ lies in the interior of convex hexagon $B_1B_2B_3B_4B_5B_6$ such that $A_1A_2 \parallel B_1B_2$, $A_2A_3 \parallel B_2B_3$, ..., $A_6A_1 \parallel B_6B_1$. Prove that the areas of simple hexagons $A_1B_2A_3B_4A_5B_6$ and $B_1A_2B_3A_4B_5A_6$ are equal. (A simple hexagon is a hexagon which does not intersect itself.)

Problem 9. Let $ABCD$ be a convex quadrilateral with right angle at the vertex C. Let $P \in CD$, such that $\angle APD = \angle BPC$ and $\angle BAP = \angle ABC$. Prove that

$$\overline{BC} = \frac{\overline{AP} + \overline{BP}}{2}.$$

Problem 10. Let $ABCD$ be a convex quadrilateral with area 3. The points M and N divide the line segment AB in three equal parts, such that $\overline{AM} = \overline{MN} = \overline{NB}$. The points P and Q divide the line segment CD in three equal parts, such that $\overline{CP} = \overline{PQ} = \overline{QD}$. Prove that the area of $MNPQ$ is 1.

Problem 11. Let $ABCD$ be a convex quadrilateral ($\overline{AB} > \overline{CD}, \overline{AD} > \overline{BC}$). Let $AB \cap DC = P$ and $AD \cap BC = Q$. If $\overline{AP} = \overline{AQ}$ and $\overline{AB} = \overline{AD}$, prove that AC is the angle bisector of $\angle BAD$.

Problem 12. Let $ABCD$ be a parallelogram. Let M and N be the midpoints of the sides BC and DA, respectively. Prove that the lines AM and CN divide the diagonal BD in three equal parts.

Problem 13. Let $ABCD$ be a parallelogram with area 1. Let M be the midpoint of the side AD. Let $BM \cap AC = P$. Find the area of $MPCD$.

Problem 14. Let O, I, H be the circumcenter, incenter and orthocenter, respectively, of $\triangle ABC$. Prove that B, C, O, I, H lie on a circle if and only if $\angle BAC = 60°$.

Problem 15. Let H and O be the orthocenter and circumcenter in a triangle ABC, respectively. If $\angle BAC = 60°$, prove that $\overline{AH} = \overline{AO}$. Is the converse true?

Problem 16 (Serbia 2017, Opstinsko IIA)**.** Let T be the centroid of a triangle ABC and let t be a line that passes through T, such that A and B are on one side of t and C is on the other side. Let A', B' and C' be the orthogonal projections of A, B and C, respectively, to the line t. Prove that $\overline{AA'} + \overline{BB'} = \overline{CC'}$.

Problem 17 (Serbia 2014, Okruzno IB)**.** Let $ABCDEF$ be a convex hexagon with $\overline{AB} = \overline{AF}$, $\overline{BC} = \overline{CD}$ and $\overline{DE} = \overline{EF}$. Prove that the angle bisectors of $\angle BAF$, $\angle BCD$ and $\angle DEF$ are concurrent.

Problem 18 (IGO 2018, Intermediate)**.** In a convex quadrilateral $ABCD$, the diagonals AC and BD meet at the point P. We know that $\angle DAC = 90°$ and $2\angle ADB = \angle ACB$. If we have $\angle DBC + 2\angle ADC = 180°$, prove that $2\overline{AP} = \overline{BP}$.

Problem 19 (Serbia 2014, Okruzno IB)**.** Let ABC be a triangle with $\angle B > \angle C$. The angle bisector of $\angle A$ intersects BC at D. The perpendicular from B to AD intersects the circumcircle of $\triangle ABD$ again at E. Prove that the circumcenter of $\triangle ABC$ lies on the line AE.

Problem 20 (Serbia 2014, Opstinski IA)**.** Let $ABCD$ be a quadrilateral such that $\angle BCA + \angle CAD = 180°$ and $\overline{AB} = \overline{AD} + \overline{BC}$. Prove that

$$\angle BAC + \angle ACD = \angle CDA.$$

Problem 21 (Serbia 2016, Okruzno IA)**.** Let $ABCD$ be a convex quadrilateral with $\overline{AD} = \overline{BC}$ and $\angle A + \angle B = 120°$. Let E be the midpoint of the side CD and let F and G be the midpoints of the diagonals AC and BD, respectively. Prove that EFG is an equilateral triangle.

Problem 22. Let ABC be a right triangle ($\angle BCA = 90°$). Let CD be the altitude from the vertex C. Prove that the distances from the point D to the legs of the triangle are proportional to the lengths of the legs.

Problem 23 (IGO 2017, Intermediate)**.** Let ABC be an acute-angled triangle with $A = 60°$. Let E, F be the feet of altitudes through B, C respectively. Prove that $\overline{CE} - \overline{BF} = \frac{3}{2}(\overline{AC} - \overline{AB})$.

Problem 24. Let ABC be a right triangle ($\angle BCA = 90°$). Let AD and BE be angle bisectors ($D \in BC, E \in CA$). Let N and M be the feet of the perpendiculars from D and E, respectively, to the hypotenuse AB. Prove that $\angle MCN = 45°$.

Problem 25 (Serbia 2018, Drzavno VI)**.** Let ABC be an acute triangle and let AX and AY be rays, such that the angles $\angle XAB$ and $\angle YAC$ have no common interior point with $\triangle ABC$ and $\angle XAB = \angle YAC < 90°$. Let B' and C' be feet of the perpendiculars from B and C to AX and AY, respectively. If M is the midpoint of BC, prove that $\overline{MB'} = \overline{MC'}$.

Problem 26 (Romania JBMO TST 2016)**.** Let ABC be an acute triangle where $\angle BAC = 60°$. Prove that if the Euler's line of $\triangle ABC$ intersects AB and AC at D and E, respectively, then $\triangle ADE$ is equilateral.

Problem 27 (Stefan Lozanovski)**.** In the triangle ABC, $\gamma = 60°$. Let O be the circumcenter of $\triangle ABC$. AO intersects BC at M and BO intersects AC at N. Prove that $\overline{AN} = \overline{BM}$.

Problem 28 (Sharygin 2011, Correspondence Round)**.** The diagonals of a trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

Problem 29 (IGO 2014, Junior)**.** The inscribed circle of $\triangle ABC$ touches BC, CA and AB at D, E and F, respectively. Denote the feet of the perpendiculars from F, E to BC by K, L, respectively. Let the second intersection of these perpendiculars with the incircle be M, N, respectively. Show that

$$\frac{P_{\triangle BMD}}{P_{\triangle CND}} = \frac{\overline{DK}}{\overline{DL}}.$$

Problem 30 (Serbia 2018, Opstinsko IIIA)**.** Let I be the incenter of a triangle ABC ($\overline{AB} < \overline{AC}$). The line AI intersects the circumcircle of ABC again at D. The circumcircle of CDI intersects BI again at K. Prove that $\overline{BK} = \overline{CK}$.

Problem 31 (IGO 2014, Junior)**.** In a right triangle ABC we have $\angle A = 90°$ and $\angle C = 30°$. Denote by ω the circle passing through A which is tangent to BC at the midpoint. Assume that ω intersects AC and the circumcircle of $\triangle ABC$ at N and M, respectively. Prove that $MN \perp BC$.

Problem 32. The angle bisectors of the adjacent angles in a quadrilateral $ABCD$ intersect at the points E, F, G and H. Prove that $EFGH$ is cyclic.

Problem 33 (JBMO Shortlist 2015)**.** Let t be the tangent at the vertex C to the circumcircle of triangle ABC. A line p parallel to t intersects BC and AC at points D and E, respectively. Prove that the points A, B, D and E are concyclic.

Problem 34. Two circles intersect at A and B. One of their common tangents touches the circles at P and Q. Let A' be the reflection of A across the line PQ. Prove that $A'PBQ$ is a cyclic quadrilateral.

Problem 35 (IGO 2016, Intermediate)**.** Let two circles C_1 and C_2 intersect in points A and B. The tangent to C_1 at A intersects C_2 in P and the line PB intersects C_1 for the second time in Q (suppose that Q is outside C_2). The tangent to C_2 from Q intersects C_1 and C_2 in C and D, respectively. (The points A and D lie on different sides of the line PQ.) Show that AD is the bisector of $\angle CAP$.

Problem 36. Let $ABCD$ be a cyclic quadrilateral and let S be the intersection of its diagonals ($\angle ASB < 90°$). If H is the orthocenter of $\triangle ABS$ and O is the circumcenter of $\triangle CDS$, prove that the points H, S and O are collinear.

Problem 37. Let $ABCD$ be a cyclic quadrilateral. The rays AB and DC intersect at P and the rays AD and BC intersect at Q. The circumcircles of $\triangle BCP$ and $\triangle CDQ$ intersect at R. Prove that the points P, Q and R are collinear.

Problem 38. The diagonals of a cyclic quadrilateral $ABCD$ intersect at S. The circumcircle of $\triangle ABS$ intersects line BC at M, and the circumcircle of $\triangle ADS$ intersects line CD at N. Prove that S, M and N are collinear.

Problem 39 (IGO 2018, Intermediate)**.** Let ω_1 and ω_2 be two circles with centers O_1 and O_2, respectively. These two circles intersect at points A and B. Line O_1B intersects ω_2 for the second time at point C, and line O_2A intersects ω_1 for the second time at point D. Let X be the second intersection of AC and ω_1 and let Y be the second intersection of BD and ω_2. Prove that $\overline{CX} = \overline{DY}$.

Problem 40. Let D, E and F be points on the sides BC, CA and AB, respectively, such that $BCEF$ is a cyclic quadrilateral. Let P be the second intersection of the circumcircles of $\triangle BDF$ and $\triangle CDE$. Prove that A, D and P are collinear.

Problem 41. Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that TP is the internal angle bisector of $\angle ATB$.

Problem 42. Let $ABCD$ be a trapezoid ($AB \parallel CD$). Let $AC \cap BD = E$ and $AD \cap BC = F$. Let M, N be midpoints of AB, CD, respectively. Prove that the points E, F, M, N are collinear.

Problem 43 (IGO 2018, Advanced)**.** In acute triangle ABC, $\angle A = 45°$, points O and H are the circumcenter and the orthocenter, respectively. The foot of the altitude from B is D. Point X is the midpoint of arc $\widehat{ADH}$ of the circumcircle of $\triangle ADH$. Prove that $\overline{DX} = \overline{DO}$.

Problem 44 (IGO 2015, Intermediate)**.** The points P, A and B lie on a circle such that A lies on the shorter arc $\widehat{PB}$. The point Q lies inside the circle such that the points A and Q are on opposite sides of the line PB, $\angle PAQ = 90°$ and $\overline{PQ} = \overline{BQ}$. Prove that $\angle AQB - \angle PQA$ is equal to the central angle over the arc $\widehat{AB}$.

Problem 45 (IGO 2018, Elementary)**.** There are two circles with centers O_1, O_2 that lie inside a circle ω and are tangent to it. Chord AB of ω is tangent to these two circles such that they lie on opposite sides of this chord. Prove that $\angle O_1AO_2 + \angle O_1BO_2 > 90°$.

Problem 46. In a triangle ABC let AD be an angle bisector ($D \in BC$). Let E and F be points on the interior segments AC and AB, respectively, such that $\angle BFD = \angle BDA$ and $\angle CED = \angle CDA$. Prove that EF is parallel to BC.

Problem 47 (Iran MO, 3rd Round, 2017)**.** Let ABC be a triangle. Suppose that X and Y are points in the plane such that BX and CY are tangent to the circumcircle of $\triangle ABC$, $\overline{AB} = \overline{BX}$, $\overline{AC} = \overline{CY}$ and X, Y and A are in the same side of BC. If I is the incenter of $\triangle ABC$, prove that $\angle BAC + \angle XIY = 180°$.

Problem 48 (IGO 2014, Junior)**.** In a triangle ABC we have $\angle C = \angle A + 90°$. The point D on the continuation of BC is given such that $\overline{AC} = \overline{AD}$. A point E lies on the opposite side of BC than A, such that $\angle EBC = \angle A$ and $\angle EDC = \frac{1}{2}\angle A$. Prove that $\angle CED = \angle ABC$.

Problem 49 (IGO 2017, Intermediate)**.** Two circles ω_1, ω_2 intersect at A, B. An arbitrary line through B meets ω_1, ω_2 at C, D, respectively. The points E, F are chosen on ω_1, ω_2, respectively, so that $\overline{CE} = \overline{CB}$ and $\overline{BD} = \overline{DF}$. Suppose that BF meets ω_1 at P and BE meets ω_2 at Q. Prove that A, P, Q are collinear.

Problem 50 (IGO 2018, Advanced)**.** Two circles ω_1 and ω_2 intersect each other at points A and B. Let PQ be a common tangent line of these two circles with $P \in \omega_1$ and $Q \in \omega_2$. An arbitrary point X lies on ω_1. Line AX intersects ω_2 for the second time at Y. Point $Y' \neq Y$ lies on ω_2 such that $\overline{QY} = \overline{QY'}$. Line $Y'B$ intersects ω_1 for the second time at X'. Prove that $\overline{PX} = \overline{PX'}$.

Problem 51 (JBMO Shortlist 2012)**.** Let ABC be an acute-angled triangle with circumcircle ω, and let O and H be the triangle's circumcenter and orthocenter, respectively. Let also A' be the point where the angle bisector of $\angle BAC$ meets ω. If $\overline{A'H} = \overline{AH}$, then find the measure of $\angle BAC$.

Problem 52 (EGMO 2012)**.** Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. Let K be the circumcenter of triangle AFE. Prove that the lines DK and BC are perpendicular.

Problem 53 (EGMO 2015)**.** Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45°$, show that CF is tangent to ω.

Problem 54 (IGO 2017, Intermediate)**.** In the isosceles $\triangle ABC$ ($\overline{AB} = \overline{AC}$), let ℓ be a line parallel to BC through A. Let D be an arbitrary point on ℓ. Let E, F be the feet of perpendiculars through A to BD, CD, respectively. Suppose that P, Q are the orthogonal projections of E, F on ℓ. Prove that $\overline{AP} + \overline{AQ} \leq \overline{AB}$.

Problem 55 (Bosnia and Herzegovina TST 2013)**.** Triangle ABC is right angled at C. Lines AM and BN are internal angle bisectors. AM and BN intersect the altitude CD at points P and Q, respectively. Prove that the line which passes through the midpoints of segments QN and PM is parallel to AB.

Problem 56. Let P be a point outside a circle ω. Let A and B be points on ω, such that PA and PB are tangents to ω. On the minor arc $\widehat{AB}$ lies an arbitrary point C. Let D, E and F be the feet of the perpendiculars from C to AB, PA and PB, respectively. Prove that $\overline{CD}^2 = \overline{CE} \cdot \overline{CF}$.

Problem 57 (Serbia 2017, Okruzno IVA)**.** Let PA and PB be the tangents from P to a circle ω $(A, B \in \omega)$. Let Q be a point on the line PA, such that A is between P and Q and $\overline{PA} = \overline{AQ}$ and let C be a point on the line segment AB. The circumcircle of $\triangle PBC$ intersects ω again at D. Prove that $\angle PBD = \angle QCA$.

Problem 58. Let C be a point on a semicircle with diameter AB and let D be the midpoint of arc AC. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

Problem 59 (IGO 2014, Senior)**.** An acute-angled triangle ABC is given. The circle ω with diameter BC intersects AB, AC at E, F, respectively. Let M be the midpoint of BC and P the intersection point of AM and EF. Let X be a point on the arc $\widehat{EF}$ of ω and Y be the second intersection point of XP and ω. Show that $\angle XAY = \angle XYM$.

Problem 60 (JBMO Shortlist 2015)**.** The point P is outside the circle Ω. Two tangent lines, passing through P touch Ω at points A and B. The median AM in the triangle ABP intersects Ω at C and PC intersects Ω again at D. Prove that $AD \parallel BP$.

Problem 61. Let k_1 and k_2 be two circles intersecting at A and B. Let t_1 and t_2 be the tangents to k_1 and k_2 at point A and let $t_1 \cap k_2 = \{A, C\}$, $t_2 \cap k_1 = \{A, D\}$. If E is a point on the ray AB, such that $\overline{AE} = 2 \cdot \overline{AB}$, prove that $ACED$ is cyclic.

Problem 62. In a triangle ABC, let $M \in BC$, $N \in AC$, $K = AM \cap BN$, such that the circumcircles of $\triangle AKN$ and $\triangle BKM$ intersect at the orthocenter H of $\triangle ABC$. Prove that $\overline{AM} = \overline{BN}$.

Problem 63. Let A_1, B_1 and C_1 be the second intersections of the angle bisectors of $\triangle ABC$ with its circumcircle. Prove that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A_1B_1C_1$.

Problem 64. Let A_1, B_1 and C_1 be the second intersections of the altitudes of $\triangle ABC$ with its circumcircle. Prove that the orthocenter of $\triangle ABC$ is the incenter of $\triangle A_1B_1C_1$.

Problem 65 (IGO 2015, Intermediate)**.** In acute-angled triangle ABC, BH is the altitude from the vertex B $(H \in AC)$. The points D and E are midpoints of AB and AC, respectively. Suppose that F is the reflection of H with respect to ED. Prove that the line BF passes through circumcenter of $\triangle ABC$.

Problem 66 (IbMO 1998)**.** The incircle of $\triangle ABC$ is tangent to the sides BC, CA and AB at D, E and F, respectively. Let AD intersect the incircle at Q. Show that the line EQ bisects the segment AF if and only if $\overline{AC} = \overline{BC}$.

Problem 67 (IGO 2014, Junior; Macedonia JMO 2017)**.** Two points X and Y lie on the arc $\widehat{BC}$ (that does not contain A) of the circumcircle of $\triangle ABC$, such that $\angle BAX = \angle CAY$. Let M be the midpoint of the chord AX. Show that $\overline{BM} + \overline{CM} > \overline{AY}$.

Problem 68 (JBMO Shortlist 2015)**.** Let ω be a circle with center O and let A and B be two points on ω that are not diametrically opposite. The bisector of $\angle ABO$ intersects ω again at C, the circumcircle of $\triangle AOB$ at K and the circumcircle of $\triangle AOC$ at L. Prove that K is the circumcenter of $\triangle AOC$ and L is the incenter of $\triangle AOB$.

Problem 69 (JBMO Shortlist 2016)**.** Let ABC be an acute triangle whose shortest side is BC. Consider a variable point P on the side BC, and let D and E be points on AB and AC, respectively, such that $\overline{BD} = \overline{BP}$ and $\overline{CP} = \overline{CE}$. Prove that, as P traces BC, the circumcircle of $\triangle ADE$ passes through a fixed point other than A.

Problem 70 (JBMO Shortlist 2010)**.** Let ABC be acute-angled triangle. A circle ω_1, centered at O_1, passes through points B and C and meets the sides AB and AC at points D and E, respectively. Let ω_2, centered at O_2, be the circumcircle of $\triangle ADE$. Prove that $\overline{O_1O_2}$ is equal to the circumradius of $\triangle ABC$.

Problem 71 (Macedonia JMO 2019, Stefan Lozanovski)**.** Circles ω_1 and ω_2 intersect at points A and B. Let t_1 and t_2 be the tangents to ω_1 and ω_2, respectively, at point A. Let the second intersection of ω_1 and t_2 be C, and let the second intersection of ω_2 and t_1 be D. Points P and E lie on the ray AB, such that B lies between A and P, and $\overline{AE} = 2 \cdot \overline{AP}$. The circumcircle of $\triangle BCE$ intersects t_2 again at point Q, whereas the circumcircle of $\triangle BDE$ intersects t_1 again at point R. Prove that points P, Q and R are collinear.

Problem 72 (IGO 2017, Advanced)**.** In triangle ABC, the incircle, with center I, touches the sides BC at point D. Line DI meets AC at X. The tangent line from X to the incircle (different from AC) intersects AB at Y. If YI and BC intersect at point Z, prove that $\overline{AB} = \overline{BZ}$.

Problem 73. Let I be the incenter and AB the shortest side of the triangle ABC. The circle centered at I passing through C intersects the ray AB in P and the ray BA in Q. Prove that the circumcircles of $\triangle CAQ$ and $\triangle CBP$ intersect at the angle bisector of $\angle ACB$.

Problem 74 (IGO 2016, Advanced)**.** Let the circles ω and ω' intersect in A and B. The tangent to circle ω at A intersects ω' in C. The tangent to circle ω' at A intersects ω in D. Suppose that CD intersects ω and ω' in E and F, respectively (assume that E is between F and C). The perpendicular to AC from E intersects ω' in point P and the perpendicular to AD from F intersects ω in point Q (The points A, P and Q lie on the same side of the line CD). Prove that the points A, P and Q are collinear.

Problem 75 (IGO 2016, Advanced)**.** In acute-angled triangle ABC, the altitude from A meets BC at D and M is the midpoint of AC. Suppose that X is a point such that $\angle AXB = \angle DXM = 90^\circ$ (assume that X and C lie on opposite sides of the line BM). Show that $\angle XMB = 2\angle MBC$.

Problem 76 (Sharygin 2017, Final Round)**.** Let O and H be the circumcenter and orthocenter of $\triangle ABC$, respectively. The perpendicular bisector of AH meets AB and AC at D and E, respectively. Show that $\angle AOD = \angle AOE$.

Problem 77 (Romania JBMO TST 2016)**.** Let O be the circumcenter of a triangle ABC. Let D, E and F be the tangent points of the $A-$excircle with the lines BC, CA and AB, respectively. If the $A-$excircle has radius equal to the circumradius of $\triangle ABC$, prove that $OD \perp EF$.

Problem 78 (Macedonia MO 2018)**.** Given is an acute $\triangle ABC$ with orthocenter H. The point H' is symmetric to H over the side AB. Let N be the intersection point of HH' and AB. The circle passing through A, N and H' intersects AC for the second time in M, and the circle passing through B, N and H' intersects BC for the second time in P. Prove that M, N and P are collinear.

Problem 79 (EMC 2019, Junior, Stefan Lozanovski)**.** Let ABC be a triangle with circumcircle ω. Let ℓ_B and ℓ_C be two lines though the points B and C, respectively, such that $\ell_B \parallel \ell_C$. The second intersections of ℓ_B and ℓ_C with ω are D and E, respectively ($D \in \widehat{AB}, E \in \widehat{AC}$). Let DA intersect ℓ_C at F and let EA intersect ℓ_B at G. If O, O_1 and O_2 are circumcenters of $\triangle ABC$, $\triangle ADG$ and $\triangle AEF$, respectively, and P is the circumcenter of $\triangle OO_1O_2$, prove that $\ell_B \parallel OP \parallel \ell_C$.

Problem 80 (IMO 2002/2)**.** The circle ω has center O, and BC is a diameter of ω. Let A be a point of ω such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets ω at E and at F. Prove that I is the incenter of $\triangle CEF$.

Problem 81 (IGO 2015, Advanced)**.** Two circles ω_1 and ω_2 (with centers O_1 and O_2, respectively) intersect at A and B. The point X lies on ω_2. Let point Y be a point on ω_1 such that $\angle XBY = 90°$. Let X' be the second point of intersection of the line O_1X and ω_2 and K be the second point of intersection of $X'Y$ and ω_2. Prove that X is the midpoint of arc $\widehat{AK}$.

Problem 82 (Serbia MO 2015)**.** Let $ABCD$ be a cyclic quadrilateral. Let M, N, P, Q be midpoints of the sides DA, AB, BC, CD, respectively. Let E be the intersection of the diagonals. Let k_1, k_2 be the circumcircles of $\triangle EMN$, $\triangle EPQ$, respectively. Let F be the second point of intersection of k_1 and k_2. Prove that $EF \perp AC$.

Problem 83. In a triangle ABC ($\overline{AB} \neq \overline{AC}$), let the incircle centered at I touch the sides BC, CA and AB at points D, E and F, respectively. Let Y and Z be the intersections of the line through A parallel to BC with the lines DF and DE, respectively. Let M and N be midpoints of DY and DZ. Prove that the quadrilateral $IMAN$ is cyclic.

Problem 84 (International Zhautykov Olympiad 2013)**.** Given a trapezoid $ABCD$ ($AD \parallel BC$) with $\angle ABC > 90°$. Point M is chosen on the lateral side AB. Let O_1 and O_2 be the circumcenters of $\triangle MAD$ and $\triangle MBC$, respectively. The circumcircles of $\triangle MO_1D$ and $\triangle MO_2C$ meet again at the point N. Prove that the line O_1O_2 passes through the point N.

Problem 85 (APMO 2018)**.** Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $\overline{FJ} = \overline{FA}$.

Problem 86. A circle ω has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides of the quadrilateral are tangent to ω. Prove that $\overline{AD} + \overline{BC} = \overline{AB}$.

Problem 87. Let BB' and CC' be altitudes in the acute-angled triangle ABC. Let M and N be points on the line segments BB' and CC', respectively, such that $\angle AMC = 90° = \angle ANB$. Prove that $\overline{AM} = \overline{AN}$.

Problem 88 (Serbia 2016, Drzavno)**.** In $\triangle ABC$, the angle bisector of $\angle BAC$ intersects BC at D. Let M be the midpoint of BD. Let k be a circle through A that is tangent to BC at D and let the second intersections of k with the lines AM and AC be P and Q, respectively. Prove that the points B, P and Q are collinear.

Problem 89. Let ABC be a triangle and let H be its orthocenter. Let M be the midpoint of BC. The perpendicular to MH through H intersects AB and AC at P and Q, respectively. Prove that $\overline{MP} = \overline{MQ}$.

Problem 90. Let F be a point inside the parallelogram $ABCD$, such that $\angle CDF = \angle CBF$. Prove that $\angle FCB = \angle FAB$.

Problem 91 (USAMO 2010)**.** Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R and S the feet of the perpendiculars from Y onto lines AX, BX, AZ and BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Problem 92 (JBMO Shortlist 2014)**.** Let ABC be a triangle such that $\overline{AB} \neq \overline{AC}$. Let M be the midpoint of BC and H be the orthocenter of $\triangle ABC$. Let D be the midpoint of AH and O the circumcenter of triangle HBC. Prove that $DAMO$ is a parallelogram.

Problem 93 (JBMO Shortlist 2016, Stefan Lozanovski)**.** Let ABC be a triangle with $\angle BAC = 60°$. Let D and E be the feet of the perpendiculars from A to the external angle bisectors at the vertices B and C in $\triangle ABC$, respectively. Let O be the circumcenter of $\triangle ABC$. Prove that the circumcircles of $\triangle ADE$ and $\triangle BOC$ are tangent to each other.

Problem 94 (JBMO Shortlist 2010)**.** In a triangle ABC, let $\angle ACB = 90°$. Let F be the foot of the altitude from C. Circle ω touches the line segment FB at point P, the altitude CF at point Q and the circumcircle of $\triangle ABC$ at point R. Prove that points A, Q, R are collinear and $\overline{AP} = \overline{AC}$.

Problem 95. Let ABC be a triangle and let H and O be its orthocenter and circumcenter, respectively. Let K be the midpoint of AH. The perpendicular to OK through K intersects AB and AC at P and Q, respectively. Prove that $\overline{OP} = \overline{OQ}$.

Problem 96 (India Postals 2015). Let $ABCD$ be a convex quadrilateral. In $\triangle ABC$, let I and J be the incenter and the $A-$excenter, respectively. In $\triangle ACD$, let K and L be the incenter and the $A-$excenter, respectively. Show that the lines IL, JK, and the bisector $\angle BCD$ are concurrent.

Problem 97 (Newton's Theorem). A circle is inscribed in a quadrilateral $ABCD$ where sides AB, BC, CD and DA touch the circle at points E, F, G and H, respectively. Then, lines AC, EG, BD and FH are concurrent.

Problem 98 (IGO 2014, Senior, modified). Let P and Q be arbitrary points on the sides AB and AC, respectively, in a triangle ABC. Let X be an arbitrary point on the line segment PQ. Two points E and F lie on AB and AC, respectively, (E and F are on the same side of PQ), such that $\angle EXP = \angle ACX$ and $\angle FXQ = \angle ABX$. If K and L denote the intersection points of EF with the circumcircle of $\triangle ABC$, show that PQ is tangent to the circumcircle of $\triangle KLX$.

Problem 99 (EGMO 2012). Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent.

Problem 100 (EGMO 2016). Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X.Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respectively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X.

Problem 101 (USAMO 1997). Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA and AB, respectively. Show that the lines through A, B and C perpendicular to EF, FD and DE, respectively, are concurrent.

Problem 102 (APMO 2020). Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Problem 103 (BMO 2016). Let $ABCD$ be a cyclic quadrilateral with $\overline{AB} < \overline{CD}$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection of the circumcircles of $\triangle MKT$ and $\triangle MLS$ lies on the segment CD.

Problem 104. Point M lies on diagonal BD of parallelogram $ABCD$. Line AM intersects side CD and line BC at points K and N, respectively. Let C_1 be the circle with center M and radius MA and C_2 be the circumcircle of $\triangle KCN$. Prove that C_1 and C_2 are orthogonal.

Problem 105 (IMO Shortlist 1996, G3). Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $\overline{CB} > \overline{CA}$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\angle FHP = \angle BAC$.

Problem 106 (AIME 2011, modified)**.** In a triangle ABC, let P and Q be the feet of the perpendiculars from C to the angle bisectors of $\angle ABC$ and $\angle CAB$, respectively. Prove that $\overline{PQ}$ is equal to the length of the tangent segment from C to the incircle of $\triangle ABC$.

Problem 107 (India MO 2010)**.** Let ABC be a triangle with circumcircle Γ. Let M be a point in the interior of $\triangle ABC$ which is also on the bisector of $\angle BAC$. Let AM, BM and CM meet Γ in A_1, B_1 and C_1, respectively. Let P be the point of intersection of A_1C_1 with AB and Q be the point of intersection of A_1B_1 with AC. Prove that $PQ \parallel BC$.

Problem 108. Let M be the midpoint of the side BC in $\triangle ABC$. Let E and F be the tangent points of the incircle and the sides CA and AB, respectively. Let the angle bisectors of $\angle B$ and $\angle C$ intersect the line EF at X and Y, respectively. Prove that $\triangle MXY$ is equilateral if and only if $\angle A = 60°$.

Problem 109. On the sides AB and AC of a triangle ABC are given points P and Q, respectively, such that $PQ \parallel BC$. Prove that the circles with diameters BQ and CP intersect on the line through A that is perpendicular to BC.

Problem 110 (APMO 2013)**.** Let ABC be an acute triangle with altitudes AD, BE, and CF and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC and OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Problem 111. Given a semicircle with diameter AB, let C and D be points on the semicircle, such that D is between A and C. Let P be the intersection of AD and BC. Prove that the value of $\overline{AP} \cdot \overline{AD} + \overline{BP} \cdot \overline{BC}$ doesn't depend on the choice of the points C and D.

Problem 112 (Singapore TST 2008, reworded)**.** Let ω and O be the circumcircle and circumcenter of right triangle ABC with $\angle B = 90°$. Let P be any point on the tangent to ω at A (other than A), and suppose ray PB intersects ω again at D. Point E lies on line CD such that $AE \parallel BC$. Prove that P, O and E are collinear.

Problem 113 (IGO 2015, Intermediate)**.** In triangle ABC, the points M, N, K are the midpoints of BC, CA, AB, respectively. Let ω_B and ω_C be two semicircles with diameters AC and AB, respectively, outside the triangle. Suppose that MK and MN intersect ω_C and ω_B at X and Y, respectively. Let the tangents at X and Y to ω_C and ω_B, respectively, intersect at Z. Prove that $AZ \perp BC$.

Problem 114 (JBMO Shortlist 2016)**.** Given an acute triangle ABC, erect triangles $\triangle ABD$ and $\triangle ACE$ externally, so that $\angle ADB = \angle AEC = 90°$ and $\angle BAD = \angle CAE$. Let $A_1 \in BC$, $B_1 \in AC$ and $C_1 \in AB$ be the feet of the altitudes in $\triangle ABC$, and let K, L be the midpoints of BC_1, CB_1, respectively. Prove that the circumcenters of $\triangle AKL$, $\triangle A_1B_1C_1$ and $\triangle DEA_1$ are collinear.

Problem 115 (All-Russian MO 2005, Round 4)**.** Let I be an incenter of ABC ($\overline{AB} < \overline{BC}$). Let M be the midpoint of AC and N be the midpoint of the arc $\widehat{ABC}$. Prove that $\angle IMA = \angle INB$.

Problem 116 (IMO 2013/4)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of $\triangle BWN$, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle $\triangle CWM$, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 117 (Open Croatian Competition in Mathematics 2011)**.** The points B and C are on the circle ω with center A. A line t is tangent to ω at a point D which lies on the smaller arc $\widehat{BC}$. The line t intersects the circumcircle of $\triangle ABC$ at points E and F. The line AE intersects BC at P. Find $\angle APD$.

Problem 118 (EGMO 2017)**.** Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcenter O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

Problem 119 (International Zhautykov Olympiad 2014)**.** Points M, N, K lie on the sides BC, CA, AB of a triangle ABC, respectively, and are different from its vertices. The triangle MNK is called *beautiful* if $\angle BAC = \angle KMN$ and $\angle ABC = \angle KNM$. If in the triangle ABC there are two beautiful triangles with a common vertex, prove that the triangle ABC is right-angled.

Problem 120 (Serbia 2016, Opstinsko IIA)**.** The incircle of ABC ($\overline{AB} < \overline{AC}$) touches the sides BC, CA and AB at D, E and F, respectively. The angle bisector of $\angle BAC$ intersects the lines DE and DF at M and N, respectively. Let K be the foot of the altitude from A to BC. Prove that D is the incenter of $\triangle MNK$.

Problem 121. Let $ABCD$ be a convex quadrilateral. Suppose that the circles with diameters AB and CD intersect at points X and Y. Let $P = AC \cap BD$ and $Q = AD \cap BC$. Prove that the points P, Q, X and Y are concyclic.

Problem 122 (Poland MO 2000)**.** Let a triangle ABC satisfy $\overline{AC} = \overline{BC}$. Let P be a point inside the triangle ABC such that $\angle PAB = \angle PBC$. Denote by M the midpoint of the segment AB. Show that $\angle APM + \angle BPC = 180°$.

Problem 123 (Macedonia MO 2015)**.** Let k_1 and k_2 be two circles that intersect at points A and B. A line through B intersects k_1 and k_2 at C and D, respectively, such that C doesn't lie inside of k_2 and D doesn't lie inside of k_1. Let M be the intersection point of the tangent lines to k_1 and k_2 that pass through C and D, respectively. Let P be the intersection of the lines AM and CD. The tangent line to k_1 passing through B intersects AD in point L. The tangent line to k_2 passing through B intersects AC in point K. Let KP intersect MD at N and LP intersect MC at Q. Prove that $MNPQ$ is a parallelogram.

Problem 124 (IMO Shortlist 2015/G3)**.** Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside $\triangle CBH$ so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Problem 125 (IGO 2016, Intermediate)**.** Let the circles ω and ω' intersect in points A and B. The tangent to circle ω at A intersects ω' at C and the tangent to circle ω' at A intersects ω at D. Suppose that the internal bisector of $\angle CAD$ intersects ω and ω' at E and F, respectively, and the external bisector of $\angle CAD$ intersects ω and ω' at X and Y, respectively. Prove that the perpendicular bisector of XY is tangent to the circumcircle of $\triangle BEF$.

Problem 126 (IbMO 2013, modified)**.** Let N be the midpoint of an arc $\widehat{XY}$ in a circle ω. Let A and B be two arbitrary points on the chord XY. The rays NA and NB intersect ω again at C and D, respectively. The tangents to ω at C and D intersect at P. The intersection of NP and XY is M. Prove that M is the midpoint of AB.

Problem 127 (Canada MO 2012)**.** Let $ABCD$ be a convex quadrilateral such that $\overline{AC} + \overline{AD} = \overline{BC} + \overline{BD}$ and let P be the point of intersection of AC and BD. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

Problem 128 (IMO 2012/1)**.** Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.
(The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 129. Let ABC be a triangle, and let the tangent points of the incircle with the sides BC, CA, AB be D, E, F, respectively. Let P, Q, R be the midpoints of BC, CA, AB, respectively. Let $PR \cap DE = K$ and $PQ \cap DF = L$. Prove that

$$\frac{\overline{BI}}{\overline{CI}} = \frac{\overline{KE}}{\overline{LF}}.$$

Problem 130. Let $ABCD$ be a cyclic quadrilateral. Prove that the intersection of the $A-$Simson line of $\triangle BCD$ with the $B-$Simson line of $\triangle ACD$ is collinear with C and the orthocenter of $\triangle ABD$.

Problem 131 (China MO 1997)**.** Let $ABCD$ be a cyclic quadrilateral. Let $AB \cap CD = P$ and $AD \cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that P, E and F are collinear.

Problem 132 (Serbia 2016, Drzavno)**.** Let $\triangle ABC$ be an acute-angled triangle with $\overline{AB} < \overline{AC}$. Let D be the midpoint of BC and let p be the reflection of the line AD with respect to the angle bisector of $\angle BAC$. If P is the foot of the perpendicular from C to the line p, prove that $\angle APD = \angle BAC$.

Problem 133 (IMO 2017/4)**.** Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc $\widehat{RS}$ of Ω so that the circumcircle Γ of $\triangle JST$ intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 134 (IMO Shortlist 2017/G3)**.** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of $\triangle PQH$ lies on a median of $\triangle ABC$.

Problem 135 (IMO 2018/1)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $\overline{AD} = \overline{AE}$. The perpendicular bisectors of BD and CE intersect minor arcs $\widehat{AB}$ and $\widehat{AC}$ of Γ at points F and G, respectively. Prove that lines DE and FG are either parallel or they coincide.

Problem 136 (IMO 2014/4)**.** Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumcircle of $\triangle ABC$.

Problem 137 (IMO 2016/1)**.** Triangle BCF has a right angle at B. Let A be the point on line CF such that $\overline{FA} = \overline{FB}$ and F lies between A and C. Point D is chosen so that $\overline{DA} = \overline{DC}$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $\overline{EA} = \overline{ED}$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Problem 138 (Navneel Singhal)**.** Let H be the orthocenter of a triangle ABC. Let M be the midpoint of BC, and let E and F be the feet of the $B-$ and $C-$altitudes onto the opposite sides. Let X be the intersection of ray MA with the circumcircle of $\triangle BHC$. Prove that HX, EF and BC are concurrent.

Problem 139 (Vietnam TST 2001)**.** Two circles intersect at A and B and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of $\triangle APQ$ intersect at S and let H be the reflection of B across the line PQ. Prove that the points A, S and H are collinear.

Problem 140. Let I be the incenter in $\triangle ABC$. The line perpendicular to BI at I intersects line AC at B'. The line perpendicular to CI at I intersects line AB at C'. The lines BB' and CC' intersect at D. Prove that $DI \perp BC$.

Problem 141 (JBMO 2002)**.** ABC is an isosceles triangle ($\overline{CA} = \overline{CB}$). Let P be a point on the arc $\widehat{AB}$ on (ABC) that doesn't contain C. Let D be the foot of the perpendicular from C to PB. Show that $\overline{PA} + \overline{PB} = 2 \cdot \overline{PD}$.

Problem 142 (JBMO 2010)**.** Let AL and BK be angle bisectors in the non-isosceles triangle ABC (L lies on the side BC, K lies on the side AC). The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $\overline{LN} = \overline{NA}$.

Problem 143 (JBMO 2013, Stefan Lozanovski)**.** Let ABC be an acute-angled triangle and let O be the center of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.

Problem 144 (JBMO 2014)**.** Consider an acute triangle ABC of area S. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocenters of the triangles MNC and MND, respectively. Find the area of the quadrilateral AH_1BH_2 in terms of S.

Problem 145 (JBMO 2015)**.** Let ABC be an acute triangle. The lines ℓ_1 and ℓ_2 are perpendicular to AB at the points A and B, respectively. The perpendicular lines from the midpoint M of AB to the lines AC and BC intersect ℓ_1 and ℓ_2 at the points E and F, respectively. If D is the intersection point of the lines EF and MC, prove that $\angle ADB = \angle EMF$.

Problem 146 (JBMO 2016)**.** A trapezoid $ABCD$ ($AB \parallel CD, \overline{AB} > \overline{CD}$) is circumscribed about a circle. The incircle of triangle ABC touches the lines AB and AC at the points M and N, respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

Problem 147 (JBMO 2017)**.** Let ABC be an acute triangle such that $\overline{AB} \neq \overline{AC}$, with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. Let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A such that $\angle BQM = \angle BCA$ and $\angle CQM = \angle CBA$. Let the line AO intersect Γ at $E \neq A$ and let the circumcircle of $\triangle ETQ$ intersect Γ at point $X \neq E$. Prove that the points A, M and X are collinear.

Problem 148 (JBMO 2018)**.** Let A', B' and C' be the reflections of the vertices of triangle ABC with respect to their opposite sides. The intersection of the circumcircles of $\triangle ABB'$ and $\triangle ACC'$ is A_1. The points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1 and CC_1 are concurrent.

Problem 149 (JBMO 2019)**.** Triangle ABC is such that $\overline{AB} < \overline{AC}$. The perpendicular bisector of side BC intersects lines AB and AC at points P and Q, respectively. Let H be the orthocenter of triangle ABC, and let M and N be the midpoints of segments BC and PQ, respectively. Prove that lines HM and AN meet on the circumcircle of ABC.

Problem 150 (Mediterranean Mathematical Competition 2018)**.** Let ABC be an acute triangle. The points E and F lie on line segment BC, such that $\angle BAE = \angle CAF$. The lines AE and AF intersect the circumcircle of $\triangle ABC$ again at M and N, respectively. The points P and Q lie on the rays AB and AC, respectively, such that $\angle AEP = \angle ABC$ and $\angle AER = \angle ACB$. Let the intersection of PR and AE be L and let the intersection of LN and BC be D. Prove that

$$\frac{1}{\overline{MN}} + \frac{1}{\overline{EF}} = \frac{1}{\overline{ED}}.$$

Problem 151 (MOP 1998)**.** Let ABC be an acute triangle and D, E and F the feet of its altitudes from A, B and C, respectively. The line through D parallel to EF meets AC and AB at Q and R, respectively. Let P be the intersection of BC and EF. Prove that the circumcircle of $\triangle PQR$ passes through the midpoint of BC.

Problem 152 (USA TST 2012)**.** In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

Problem 153. The diagonals of the quadrilateral $ABCD$ intersect at P. Let O_1 and O_2 be the circumcenters of $\triangle APD$ and $\triangle BPC$, respectively. Let M, N and O be the midpoints of AC, BD and O_1O_2, respectively. Prove that O is the circumcenter of $\triangle MPN$.

Problem 154 (JBMO Shortlist 2013)**.** Let P and Q be the midpoints of the sides BC and CD, respectively in a rectangle $ABCD$. Let K and M be the intersections of the line PD with the lines QB and QA, respectively, and let N be the intersection of the lines PA and QB. Let X, Y and Z be the midpoints of the segments AN, KN and AM, respectively. Let ℓ_1 be the line passing through X and perpendicular to MK, ℓ_2 be the line passing through Y and perpendicular to AM and ℓ_3 the line passing through Z and perpendicular to KN. Prove that the lines ℓ_1, ℓ_2 and ℓ_3 are concurrent.

Problem 155 (EMC 2012, Senior)**.** Let ABC be an acute triangle with orthocenter H. AH and CH intersect BC and AB in points A_1 and C_1, respectively. BH and A_1C_1 meet at point D. Let P be the midpoint of the segment BH. Let D' be the reflection of the point D with respect to AC. Prove that the quadrilateral $APCD'$ is cyclic.

Problem 156 (Sharygin 2018, Correspondence Round)**.** Let I be the incenter of a non-isosceles triangle ABC. Prove that there exists a unique pair of points M, N lying on the sides AC, BC, respectively, such that $\angle AIM = \angle BIN$ and $MN \parallel AB$.

Problem 157 (EGMO 2018)**.** Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

Problem 158 (CMC 2020)**.** Let ABC be a triangle and let D be a variable point on the line segment BC. Let E be the point on the circumcircle of $\triangle ABC$ lying on the opposite side of BC from A such that $\angle BAE = \angle DAC$. Let I be the incenter of $\triangle ABD$ and let J be the incenter of $\triangle ACE$. Prove that the line IJ passes through a fixed point, that is independent of D.

Problem 159 (BMO 2010)**.** Let ABC be an acute triangle with orthocentre H, and let M be the midpoint of AC. The point C_1 on AB is such that CC_1 is an altitude of the triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcentre of triangle PQR is the midpoint of the segment MM_1. Prove that M_1 lies on the segment BH_1.

Problem 160 (Macedonia MO 2009, corrected)**.** Let I be the incenter of $\triangle ABC$. Points K and L are the intersection points of the circumcircles of $\triangle BIC$ and $\triangle AIC$ with the bisectors of $\angle BAC$ and $\angle ABC$, respectively ($K, L \neq I$). Let P be the midpoint of the segment KL. Let M be the reflection of I with respect to P and N be the reflection of I with respect to C. Prove that the points K, L, M and N lie on the same circle.

Problem 161 (Macedonia MO 2016, modified)**.** Let K be the midpoint of a given segment AB. Let C be a point that doesn't lie on the line AB. Let N be the intersection of AC and the line passing through B and the midpoint of CK. Let U be the intersection point of AB and the line passing through C and $L$the midpoint of BN. Prove that the ratio of the areas of $\triangle CNL$ and $\triangle BUL$ does not depend on the choice of the point C.

Problem 162 (Morocco 2015)**.** Let ABC be a triangle and O be its circumcenter. Let T be the intersection of the circle through A and C tangent to AB and the circumcircle of $\triangle BOC$. Let K be the intersection of the lines TO and BC. Prove that KA is tangent to the circumcircle of $\triangle ABC$.

Problem 163 (IGO 2015, Advanced)**.** Let H be the orthocenter of the triangle ABC. Let ℓ_1 and ℓ_2 be two lines passing through H and perpendicular to each other. Let ℓ_1 intersects BC and the extension of AB at D and Z, respectively. Let ℓ_2 intersects BC and the extension of AC at E and X, respectively. Let Y be a point such that $YD \parallel AC$ and $YE \parallel AB$. Prove that X, Y and Z are collinear.

Problem 164 (APMO 2015)**.** Let ABC be a triangle, and let D be a point on the side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of $\triangle BXD$ intersects the circumcircle ω of $\triangle ABC$ again at point Z distinct from point B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $\overline{AB} = \overline{VW}$.

Problem 165 (APMO 2017)**.** Let ABC be a triangle with $\overline{AB} < \overline{AC}$. Let D be the intersection point of the internal bisector of $\angle BAC$ and the circumcircle of $\triangle ABC$. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of $\triangle ADZ$.

Problem 166 (U sviti matematyky, P201)**.** Triangle ABC is inscribed into the circle ω. The circle ω_1 touches the circle ω internally and touches sides AB and AC in the points M and N, respectively. The circle ω_2 also touches the circle ω internally and touches sides AB and BC in the points P and K, respectively. Prove that $NKMP$ is a parallelogram.

Problem 167 (Rioplatense MO 2013, Level 3)**.** Let ABC be an acute-angled scalene triangle, with centroid G and orthocenter H. The circle with diameter AH cuts the circumcircle of BHC at A', distinct from H. Analogously define B' and C'. Prove that A', B', C' and G are concyclic.

Problem 168. Let P and Q be isogonal conjugates with respect to $\triangle ABC$. Let $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ be their respective pedal triangles. Let $X_1 = P_2Q_3 \cap P_3Q_2$, $X_2 = P_1Q_3 \cap P_3Q_1$ and $X_3 = P_1Q_2 \cap P_2Q_1$. Prove that the points X_1, X_2 and X_3 lie on the line PQ.

Problem 169 (RMM 2018)**.** Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonal AC meets the segment DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of triangles BKP and CLQ are tangent.

Problem 170 (JBMO Shortlist 2015)**.** Let ABC be an acute triangle with $\overline{AB} \neq \overline{AC}$. The incircle ω of the triangle touches the sides BC, CA and AB at points D, E and F, respectively. The perpendicular line erected at C onto BC meets EF at M and similarly, the perpendicular line erected at B onto BC meets EF at N. The line DM meets ω again at P and the line DN meets ω again at Q. Prove that $\overline{DP} = \overline{DQ}$.

Problem 171 (IGO 2018, Advanced)**.** Let $ABCD$ be a cyclic quadrilateral. A circle passing through A and B is tangent to segment CD at point E. Another circle passing through C and D is tangent to AB at point F. Point G is the intersection point of AE and DF. Point H is the intersection point of BE and CF. Prove that the incenters of $\triangle AGF$, $\triangle BHF$, $\triangle CHE$ and $\triangle DGE$ lie on a circle.

Problem 172 (MEMO 2016, Team)**.** Let ABC be an acute triangle, $\overline{AB} \neq \overline{AC}$, and let O be its circumcenter. Line AO meets the circumcircle of $\triangle ABC$ again in D, and the line BC in E. The circumcircle of $\triangle CDE$ meets the line CA again in P. The lines PE and AB intersect in Q. Line passing through O parallel to the line PE intersects the $A-$altitude of $\triangle ABC$ in F. Prove that $\overline{FP} = \overline{FQ}$.

Problem 173 (IMO Shortlist 2006/G4)**.** Let D be a point on the side AC of $\triangle ABC$ ($\angle C < \angle A < 90°$) such that $\overline{AB} = \overline{BD}$. The incircle of $\triangle ABC$ touches AB, AC at K, L, respectively. Let J be the incenter of $\triangle BCD$. Prove that KL bisects AJ.

Problem 174. Let O be the circumcenter of the acute triangle ABC. Let D be the foot of the altitude from A. A circle ω_A has center on AD, passes through A and touches (OBC) externally at T. Prove that AT is the $A-$symmedian in $\triangle ABC$.

Problem 175 (IGO 2016, Advanced)**.** In a convex quadrilateral $ABCD$, let P be the intersection point of AD and BC. Suppose that I_1 and I_2 are the incenters of $\triangle PAB$ and $\triangle PDC$, respectively. Let O be the circumcenter of $\triangle PAB$ and H the orthocenter of $\triangle PDC$. Show that the circumcircles of $\triangle AI_1B$ and $\triangle DHC$ are tangent if and only if the circumcircles of $\triangle AOB$ and $\triangle DI_2C$ are tangent.

Problem 176 (IGO 2017, Intermediate)**.** Let X, Y be two points on the side BC of triangle ABC such that $2\overline{XY} = \overline{BC}$ (X is between B and Y). Let AA' be the diameter of the circumcircle of $\triangle AXY$. Let P be the point where AX meets the perpendicular from B to BC, and Q be the point where AY meets the perpendicular from C to BC. Prove that the tangent line from A' to the circumcircle of $\triangle AXY$ passes through the circumcenter of $\triangle APQ$.

Problem 177 (Hong Kong TST 2003)**.** In the triangle ABC, the point M is the midpoint of AC and D is a point on AB. BM and CD meet at O, with $\overline{AB} = \overline{CO}$. Prove that AB is perpendicular to BC if and only if $ADOM$ is a cyclic quadrilateral.

Problem 178 (Stefan Lozanovski)**.** Let D be a point on the side AB in $\triangle ABC$. Let F be a point on CD such that $\overline{AB} = \overline{CF}$. The circumcircle of $\triangle BDF$ intersects BC again at E. Assume that A, F and E are collinear. If $\angle ACB = \gamma$, find the measurement of $\angle ADC$.

Problem 179 (Stefan Lozanovski)**.** Let AA' be a median in the triangle ABC. Let D be a point on AA' and let the intersection of BD and AC be E. The circumcircle of $\triangle BCE$ intersects AB again at F. If C, D and F are collinear, prove that $\triangle ABC$ is isosceles.

Problem 180 (IGO 2016, Intermediate)**.** Let ω be the circumcircle of right-angled triangle ABC ($\angle A = 90°$). The tangent to ω at point A intersects the line BC at point P. Suppose that M is the midpoint of the minor arc $\widehat{AB}$, and PM intersects ω for the second time in Q. The tangent to ω at point Q intersects AC at K. Prove that $\angle PKC = 90°$.

Problem 181 (EGMO 2016)**.** Two circles ω_1 and ω_2 with equal radii intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

Problem 182 (USAMO 2012)**.** Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

Problem 183 (APMO 2019)**.** Let ABC be a scalene triangle with circumcircle Γ. Let M be the midpoint of BC. A variable point P lies on the line segment AM. The circumcircles of $\triangle BPM$ and $\triangle CPM$ intersect Γ again at points D and E, respectively. The lines DP and EP intersect the circumcircles of $\triangle CPM$ and $\triangle BPM$ for the second time at X and Y, respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A.

Problem 184 (USAMO 2008)**.** Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E, respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Problem 185 (APMO 2016)**.** We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90°$ and $\overline{AB} = \overline{AC}$.

Problem 186 (Serbia MO 2017)**.** Let $ABCD$ be a convex cyclic quadrilateral. The lines AD and BC intersect at E. Let M and N be points on the sides AD and BC, respectively, such that $\frac{\overline{AM}}{\overline{MD}} = \frac{\overline{BN}}{\overline{NC}}$. The circumcircles of $\triangle EMN$ and $ABCD$ intersect at X and Y. Prove that the lines AB, CD and XY are concurrent or all parallel.

Problem 187 (Russia 2003)**.** Let ABC be a triangle with $AB \neq AC$. Point E is such that $\overline{AE} = \overline{BE}$ and $BE \perp BC$. Point F is such that $\overline{AF} = \overline{CF}$ and $CF \perp BC$. Let D be the point on line BC such that AD is tangent to the circumcircle of $\triangle ABC$. Prove that D, E and F are collinear.

Problem 188 (IbMO 2015)**.** Let ABC be an acute triangle and let D be the foot of the perpendicular from A to side BC. Let P be a point on segment AD. Lines BP and CP intersect sides AC and AB at E and F, respectively. Let J and K be the feet of the perpendiculars from E and F, respectively, to AD. Show that

$$\frac{\overline{FK}}{\overline{KD}} = \frac{\overline{EJ}}{\overline{JD}}.$$

Problem 189 (BMO 2009)**.** Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of $\triangle BMP$ and $\triangle CNP$ meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

Problem 190 (RMM 2015/4)**.** Let ABC be a triangle, and let D be the point where the incircle touches the side BC. Let I_B and I_C be the incentres of the triangles $\triangle ABD$ and $\triangle ACD$, respectively. Prove that the circumcentre of $\triangle AI_BI_C$ lies on the angle bisector of $\angle BAC$.

Problem 191 (Stefan Lozanovski)**.** Let S be a point on AC, such that BS is an angle bisector in the triangle ABC. Let O_1 and O_2 be the circumcenters of $\triangle ABS$ and $\triangle BSC$, respectively. The median AM in $\triangle ABC$ intersects BS at X. Prove that the lines AB, O_1O_2 and CX are concurrent.

Problem 192 (Simulation of Croatia MO 2019)**.** Let $ABCD$ be a convex quadrilateral such that $AD \nparallel BC$. Let E be the intersection of the diagonals and let F be the intersection of the lines AD and BC. Assume that there exists a point P inside $ABCD$ such that the projections from P to the sides of $ABCD$ are vertices of a rectangle. Let ω be the circumcircle of that rectangle. Let ω intersect AD and BC again at M and N, respectively. Let G be the intersection of the tangents to ω at M and N. Prove that $G \in EF$.

Problem 193 (USA JMO 2014)**.** Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides BC, CA, AB and let E, F be the tangent points of γ with CA, AB, respectively. Let U, V be the intersections of line EF with lines MN, MP, respectively, and let X be the midpoint of arc BAC of Γ. Prove that XI bisects UV.

Problem 194 (China MO 1992)**.** A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC, BD of $ABCD$ meet at P. Circumcircles of $\triangle ABP$ and $\triangle CDP$ meet at P and Q (O, P and Q are pairwise distinct). Show that $\angle OQP = 90°$.

Problem 195 (BMO 2017). Consider an acute-angled triangle ABC with $\overline{AB} < \overline{AC}$ and let ω be its circumscribed circle. Let t_B and t_C be the tangents to the circle ω at points B and C, respectively, and let L be their intersection. The line through B parallel to AC intersects t_C at D. The line through C parallel to AB intersects t_B at E. The circumcircle of the triangle BDC intersects AC in T, where T is located between A and C. The circumcircle of the triangle BEC intersects the line AB in S, where B is located between S and A. Prove that ST, AL, and BC are concurrent.

Problem 196 (IMO 1983/2). Let A be one of the two points of intersection of the circles ω_1 and ω_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches ω_1 at P_1 and ω_2 at P_2, while the other touches ω_1 at Q_1 and ω_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

Problem 197 (Macedonia MO 2008). ABC is an acute-angled triangle ($\overline{AB} \neq \overline{BC}$). Let AV and AD be the angle bisector and the altitude from vertex A, respectively. The circumcircle of $\triangle AVD$ intersects CA and AB in points E and F, respectively. Prove that AD, BE and CF are concurrent.

Problem 198 (Taiwan TST2, 2014). Let P be a point inside triangle ABC, and suppose lines AP, BP, CP meet the circumcircle again at T, S, R, respectively. Let U be any point in the interior of PT. A line through U parallel to AB meets CR at W, and the line through U parallel to AC meets BS at V. Finally, the line through B parallel to CP and the line through C parallel to BP intersect at point Q. Given that RS and VW are parallel, prove that $\angle CAP = \angle BAQ$.

Problem 199 (Russia MO 1999). A circle through vertices A and B of triangle ABC meets the side BC again at D. A circle through B and C meets the side AB at E and the first circle again at F. Prove that if the points A, E, D and C lie on a circle with center O then $\angle BFO = 90°$.

Problem 200 (BMO 2018). A quadrilateral $ABCD$ is inscribed in a circle k where $\overline{AB} > \overline{CD}$ and AB is not parallel to CD. Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E. If EM bisects the angle CED prove that AB is diameter of k.

Problem 201 (BMO 2019). Let ABC be an acute scalene triangle. Let X and Y be two distinct interior points of the segment BC such that $\angle CAX = \angle YAB$. Let K, S be the feet of the perpendiculars from B to the lines AX, AY, respectively and let T, L be the feet of the perpendiculars from C to the lines AX, AY, respectively. Prove that KL and ST intersect on the line BC.

Problem 202 (IMO 2005/5). Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD, respectively, such that $\overline{BE} = \overline{DF}$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

Problem 203 (Israel MO 1995). Let ω be a semicircle with diameter PQ. A circle k is tangent internally to ω and to the segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on the segment CQ. Show that AC bisects $\angle PAB$.

Problem 204 (IMO 2007/4). In triangle ABC, the bisector of $\angle BCA$ intersects the circumcircle of $\triangle ABC$ again at R, the perpendicular bisector of BC at P and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles $\triangle RPK$ and $\triangle RQL$ have the same area.

Problem 205 (IMO 2003/4). Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB, respectively. Show that $\overline{PQ} = \overline{QR}$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Problem 206 (APMO 2013). Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Problem 207 (MEMO 2014, Team). Let the incircle k of the triangle ABC touch its side BC at D. Let the line AD intersect k at $L \neq D$ and denote the excentre of ABC opposite to A by K. Let M and N be the midpoints of BC and KM, respectively. Prove that the points B, C, N, and L are concyclic.

Problem 208 (Serbia MO 2017). Let k be the circumcircle of $\triangle ABC$ and let k_a be its $A-$excircle. Let the two common tangents of k and k_a intersect BC at P and Q. Prove that $\angle PAB = \angle CAQ$.

Problem 209 (EGMO 2013). Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$.

Problem 210 (Turkey EGMO TST 2016). Let X be a variable point on the side BC of a triangle ABC. Let B' and C' be points on the rays XB and XC, respectively, satisfying $\overline{B'X} = \overline{BC} = \overline{C'X}$. The line passing through X and parallel to AB' intersects the line AC at Y, and the line passing through X and parallel to AC' intersects the line AB at Z. Prove that all lines YZ pass through a fixed point as X varies on the line segment BC.

Problem 211 (Serbia MO 2018). Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the tangent point of the incircle with BC, prove that $\angle PDQ = 90°$.

Problem 212 (IMO Shortlist 2002/G7). The incircle of a triangle ABC touches its side BC at K. Let M be the midpoint of the altitude AD of triangle ABC. The line MK meets the incircle of triangle ABC at a point N (apart from K). Show that the circumcircle of triangle BNC is tangent to the incircle of triangle ABC at the point N.

Problem 213. The incircle of $\triangle ABC$ touches BC, CA and AB at D, E and F, respectively. The A-excircle touches BC, CA and AB at D_1, E_1 and F_1, respectively. Let $K = FD \cap E_1D_1$. Prove that $AK \perp BC$.

Problem 214 (IMO Shortlist 2007/G3)**.** The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that the line CD separates the points P and Q and $\angle AQD = \angle CQB$. Prove that $\angle BQP = \angle DAQ$.

Problem 215. Let ω and I be the circumcircle and incenter of $\triangle ABC$. A circle Ω is internally tangent to ω and is also tangent to AB, AC at P, Q, respectively. The tangent of Ω which is parallel to BC intersects AB, AC at D, E, respectively. Let J be the intersection of BE and CD, and let K be the intersection of BQ and CP. Prove that the points I, J, K are collinear.

Problem 216 (Vietnam TST 2003)**.** Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

Problem 217 (IMO Shortlist 2004/G7)**.** For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

Problem 218 (IMO Shortlist 2004/G8)**.** In a cyclic quadrilateral $ABCD$, let E be the intersection of AD and BC (so that C is between B and E), and F be the intersection of AC and BD. Let M be the midpoint of CD, and $N \neq M$ be a point on the circumcircle of $\triangle ABM$ such that $\frac{\overline{AM}}{\overline{MB}} = \frac{\overline{AN}}{\overline{NB}}$. Show that E, F and N are collinear.

Problem 219 (Peru TST for IberoAmerican MO 2014)**.** The incircle of $\triangle ABC$, centered at I, touches AC and AB at E and F, respectively. Let H be the foot of the altitude from A and let $R = CI \cap AH$ and $Q = BI \cap AH$. Prove that the midpoint of AH lies on the radical axis of (REC) and (QFB).

Problem 220 (Serbia MO 2016)**.** Let ABC be a triangle and I its incenter. Let M be the midpoint of BC and D the tangent point of the incircle and BC. Prove that the perpendiculars from M, D and A to AI, IM and BC, respectively are concurrent.

Problem 221 (Mathematical Reflections)**.** Let D, E, F on BC, CA, AB be the touch points of the incircle of $\triangle ABC$. Line EF intersects (ABC) at X_1, X_2. The incircle of $\triangle ABC$ and (DX_1X_2) intersect again at Y. If T is the tangent point of the $A-$mixtilinear incircle and (ABC), prove that A, Y, T are collinear.

Problem 222 (Iran TST 2009)**.** In triangle ABC, D, E and F are the points of tangency of the incircle (centered at I) to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $\overline{DP} = \overline{MP}$. If H is the orthocenter of $\triangle BIC$, prove that PH bisects EF.

Problem 223 (Bulgaria MO 2014)**.** Let $ABCD$ be a quadrilateral inscribed in a circle ω. The diagonals AC and BD meet at E. The rays CB and DA meet at F. Prove that the line through the incenters of $\triangle ABE$ and $\triangle ABF$ and the line through the incenters of $\triangle CDE$ and $\triangle CDF$ meet at a point lying on ω.

Problem 224 (IGO 2016, Advanced)**.** In a convex quadrilateral $ABCD$, the lines AB and CD meet at point E and the lines AD and BC meet at point F. Let P be the intersection of the diagonals AC and BD. Suppose that ω_1 is a circle passing through D and tangent to AC at P. Also suppose that ω_2 is a circle passing through C and tangent to BD at P. Let X be the intersection point of ω_1 and AD, and Y be the intersection point of ω_2 and BC. Suppose that the circles ω_1 and ω_2 intersect for the second time at Q. Prove that the perpendicular from P to the line EF passes through the circumcenter of $\triangle XQY$.

Problem 225 (Turkey MO 2015)**.** In a cyclic quadrilateral $ABCD$ whose largest interior angle is D, lines BC and AD intersect at point E, while lines AB and CD intersect at point F. A point P is taken in the interior of quadrilateral $ABCD$ for which $\angle EPD = \angle FPD = \angle BAD$. O is the circumcenter of quadrilateral $ABCD$. Line FO intersects the lines AD, EP, BC at X, Q, Y, respectively. If $\angle DQX = \angle CQY$, show that $\angle AEB = 90°$.

Problem 226 (USA TST 2015)**.** Let ABC be a triangle ($\overline{AB} < \overline{AC}$) with incenter I whose incircle is tangent to BC, CA, AB at D, E, F, respectively. Denote by M the midpoint of BC. Let Q be a point on the incircle such that $\angle AQD = 90°$. Let P be the point inside the triangle on line AI for which $\overline{MD} = \overline{MP}$. Prove that $\angle PQE = 90°$.

Problem 227. Let the incircle and the $A-$mixtilinear incircle of a triangle ABC touch AC, AB at E, F and K, J, respectively. Lines EF and JK meet BC at X and Y, respectively. The $A-$mixtilinear incircle touches the circumcircle of ABC at T and the reflection of A in O, the circumcenter, is A'. The midpoint of arc BAC is M. Prove that the lines TA', OY, MX are concurrent.

Problem 228 (Serbia MO 2016)**.** Let ABC be a triangle and O be its circumcenter. A line tangent to the circumcircle of the triangle BOC intersects sides AB at D and AC at E. Let A' be the image of A with respect to the line DE. Prove that the circumcircle of $\triangle A'DE$ is tangent to the circumcircle of $\triangle ABC$.

Problem 229 (IMO 2008/6)**.** Let $ABCD$ be a convex quadrilateral ($\overline{BA} \neq \overline{BC}$). Denote the incircles of triangles $\triangle ABC$ and $\triangle ADC$ by ω_1 and ω_2, respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Problem 230 (IMO 2019/6)**.** Let I be the incenter of acute triangle ABC with $\overline{AB} \neq \overline{AC}$. The incircle ω of ABC is tangent to sides BC, CA and AB at D, E and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q. Prove that lines DI and PQ meet on the line through A perpendicular to AI.

In case you solved all the problems from a previous version, here is a list of the new problems added in each of the later versions:

Problems added in **v1.1**:
1, 2, 3, 7, 9, 11, 16, 17, 19, 20, 21, 22, 24, 30, 57, 85, 106, 108, 110, 115, 120, 127, 128, 129, 146, 164, 169, 185, 186, 193, 195, 200, 208, 211, 217, 225 and 226.

Problems added in **v1.2**:
15, 25, 26, 32, 33, 36, 37, 38, 40, 41, 46, 47, 51, 52, 53, 60, 62, 68, 73, 77, 78, 83, 87, 91, 92, 99, 100, 101, 105, 109, 111, 118, 130, 145, 147, 157, 161, 167, 170, 172, 181, 207, 212, 213, 216, 219 and 222.

Problems added in **v1.3**:
4, 8, 18, 23, 29, 31, 35, 39, 43, 44, 48, 49, 50, 54, 59, 65, 67, 70, 72, 74, 75, 76, 81, 84, 86, 94, 98, 113, 119, 125, 135, 142, 148, 153, 162, 163, 166, 171, 174, 175, 176, 180, 209, 221, 223, 224 and 227.

Problems added in **v1.4**:
14, 28, 42, 45, 66, 69, 71, 79, 80, 82, 89, 90, 93, 95, 96, 97, 102, 103, 104, 112, 114, 117, 121, 124, 126, 133, 134, 138, 140, 149, 150, 151, 152, 154, 156, 158, 165, 168, 173, 182, 183, 184, 188, 192, 198, 201, 202, 206, 210, 215, 218 and 230.

Solutions

Here is a link to an AoPS forum for discussing the solutions of the problems in this book: artofproblemsolving.com/community/c1202461.

A big thank you to user geometry6 for creating the forum and everyone else that contributed with adding problems or solutions.

Appendix A

Contests Abbreviations

Here is a list of all the abbreviated mathematical contests mentioned in this book.

Abbreviation	Full Name
MO	Mathematical Olympiad
JMO	Junior Mathematical Olympiad
IMO	International Mathematical Olympiad
TST	Team Selection Test (unless otherwise noted, for the IMO team)
BMO	Balkan Mathematical Olympiad
JBMO	Junior Balkan Mathematical Olympiad
APMO	Asian Pacific Mathematics Olympiad
EGMO	European Girls' Mathematical Olympiad
MEMO	Middle European Mathematical Olympiad
RMM	Romanian Master of Mathematics
Sharygin	Geometrical Olympiad in Honour of I. F. Sharygin
IGO	Iranian Geometry Olympiad
EMC	European Mathematical Cup
IbMO	Iberoamerican Mathematical Olympiad
AIME	American Invitational Mathematics Examination
MOP	Mathematical Olympiad Summer Program in USA
ELMO	ELMO is an annual math olympiad that happens at MOP
CMC	Cyberspace Mathematical Competition

Bibliography

[1] A. Mudgal and G. Handa, "A special point on the median."

[2] R. Olšák, "Polar duality in olympiad geometry," *Mathematical Reflections*, no. 3, 2020.

[3] B. Miladinovikj, *Zbirka zadaci po matematika za podgotvuvanje na natprevari*. Prosevetno Delo, 1999.

[4] V. Stojanovic, *Mathematiskop 3, Zbirka resenih zadataka za prvi razred srednjih skole*. IP Matematiskop, 6 ed., 2000.

[5] K. S. Kedlaya, "Geometry unbound." version of 17 Jan 2006.

[6] Y. Zhao, "Circles." IMO Training 2008.

[7] Y. Zhao, "Power of a point." Trinity Training 2011.

[8] K. Y. Li, "Famous geometry theorems," *Mathematical Excalibur*, vol. 10, no. 3, 2005.

[9] K. Y. Li, "Homothety," *Mathematical Excalibur*, vol. 9, no. 4, 2004.

[10] "Homothetic center." https://en.wikipedia.org/wiki/Homothetic_center. Accessed: 2016-09-01.

[11] K. Y. Li, "Inversion," *Mathematical Excalibur*, vol. 9, no. 2, 2004.

[12] Z. Stankova-Frenkel, "Inversion in the plane, part 1." Berkeley Math Circle 1998-99.

[13] D. Djukic, "Inversion." The IMO Compendium Group, Olympiad Training Materials.

[14] K. Y. Li, "Pole and polar," *Mathematical Excalibur*, vol. 11, no. 4, 2006.

[15] Y. Zhao, "Lemmas in euclidean geometry." IMO Training 2007.

[16] Y. Zhao, "Cyclic quadrilaterals - the big picture." Winter Camp 2009.

[17] D. Grinberg, "On cyclic quadrilaterals and the butterfly theorem." Version 16 February 2007.

[18] E. Chen, "A guessing game: Mixtilinear incircles." August 11, 2015.

[19] J.-L. Ayme, “Sawayama and Thebault’s theorem,” *Forum Geometricorum*, vol. 3, 2003. 225 - 229.

[20] J. Kwan, “Two important lemmas in olympiad geometry.” October 9, 2017.

[21] N. Rapanos, “The complete quadrilateral and its properties.” HMS - Preparation Notes for IMO 2009.

[22] N. Rapanos, “The harmonic quadrilateral and its properties.” HMS - Preparation Notes for IMO 2009.

[23] C. Pohoata, “Harmonic division and its applications,” *Mathematical Reflections*, no. 4, 2007.

[24] A. Remorov, “Projective geometry.” IMO Training 2010.

[25] D. Grinberg, “Isogonal conjugation with respect to a triangle.” version 23 September 2006.

[26] N. Singhal, “Isogonal conjugates.” October 9, 2016.

[27] “AoPS blog: Elementary, post: Isogonal line lemma (updated).” https://artofproblemsolving.com/community/c2771h1181729_isogonal_line_lemma_updated. Accessed: 2020-07-14.

[28] “AoPS blog: Liberator’s blog, post: Parallelogram isogonality lemma.” https://artofproblemsolving.com/community/c3103h1226690_parallelogram_isogonality_lemma. Accessed: 2020-07-14.

[29] T. A. Moon, “The Apollonian circles and isodynamic points,” *Mathematical Reflections*, no. 6, 2010.

[30] N. Altshiller, “On the circles of Apollonius,” *The American Mathematical Monthly*, vol. 22, no. 8, pp. 261–263, Oct. 1915.

[31] “Cut The Knot website, post: Pappus’ theorem.” https://www.cut-the-knot.org/pythagoras/Pappus.shtml. Accessed: 2020-07-24.

[32] “Mathematical society of Serbia: High school competitions.” https://dms.rs/matematika-srednje-skole/. Accessed: 2018-07-12.